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14. ABSTRACT

Supply and demand in the World oil market are balanced through responses to price movement with considerable complexity in the evolution of underlying supply-demand expectation process. In order to be able to understand the price balancing process, it is important to know the economic forces and the behavior of energy commodity spot price processes. The relationship between the different energy sources and its utility together with uncertainty also play a role in many important energy issues. The qualitative and quantitative behavior of energy commodities in which the trend in

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Stochastic Modeling and Analysis of Energy Commodity Spot Price Processes

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Below we list the main components of the research carried out in this dissertation.
(1) Employing basic economic principles, interconnected deterministic and stochastic models of linear log-spot and expected log-spot price processes coupled with non-linear volatility process are initiated. (2) Closed form solutions of the models are analyzed. (3) Introducing a change of probability measure, a risk-neutral interconnected stochastic model is derived. (4) Furthermore, under the risk-neutral measure, expectation of the square of volatility is reduced to a continuoustime deterministic delay differential equation. (5) The by-product of this exhibits the hereditary effects on the mean-square volatility process. (6) Using a numerical scheme, a time-series model is developed and utilized to estimate the state and parameters of the dynamic model. In fact, the developed time-series model includes the extended GARCH model as special case. (7) Using the Henry Hub natural gas data set, the usefulness of the linear interconnected stochastic models is outlined.

(8) Using natural and basic economic ideas, interconnected deterministic and stochastic models in (1) are extended to non-linear log-spot price, expected log-spot price and volatility processes. (9) The presented extended models are validated. (10) Closed form solution and risk-neutral models of (8) are outlined. (11) To exhibit the usefulness of the non-linear interconnected stochastic model, to increase the efficiency and to reduce the magnitude of error, it was essential to develop a modified version of extended Kalman filtering approach. The modified approach exhibits the reduction of magnitude of error. Furthermore, Henry Hub natural gas data set is used to show the advantages of the non-linear interconnected stochastic model.

(12) Parameter and state estimation problems of continuous time non-linear stochastic dynamic process is motivated to initiate an alternative innovative approach. This led to introduce the concept of statistic processes, namely, local sample mean and sample variance. (13) Then it led to the development of an interconnected discrete-time dynamic system of local statistic processes and (14) its mathematical model. (15) This paved the way for developing an innovative approach referred as Local Lagged adapted Generalized Method of Moments (LLGMM). This approach exhibits the balance between model specification and model prescription of continuous time dynamic processes. (16) In addition, it motivated to initiate conceptual computational state and parameter estimation and simulation schemes that generates a mean square sub-optimal procedure. (17) The usefulness of this approach is illustrated by applying this technique to four energy commodity data sets, the U. S. Treasury Bill Yield Interest Rate and the U.S. Eurocurrency Exchange Rate data sets for state and parameter estimation problems. (18) Moreover, the forecasting and confidence-interval problems are also investigated.

(19) The non-linear interconnected stochastic model (8) was further extended to multivariate interconnected energy commodities and sources with and without external random intervention processes. (20) Moreover, it was essential to extend the interconnected discrete-time dynamic system of local sample mean and variance processes to multivariate discrete-time dynamic system. (21) Extending the LLGMM approach in (15) to a multivariate interconnected stochastic dynamic model under intervention process, the parameters in the multivariate model are estimated. These estimated parameters help in analyzing the short and long term relationship between the energy commodities.

These developed results are applied to the Henry Hub natural gas, crude oil and coal data sets.

Stochastic Modeling and Analysis of Energy Commodity Spot Price Processes

by

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of the requirements for the degree of
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Dedication

To God.

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Abstract

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Below we list the main components of the research carried out in this dissertation.

(**1**) Employing basic economic principles, interconnected deterministic and stochastic models of linear log-spot and expected log-spot price processes coupled with non-linear volatility process are initiated. (**2**) Closed form solutions of the models are analyzed. (**3**) Introducing a change of probability measure, a risk-neutral interconnected stochastic model is derived. (**4**) Furthermore, under the risk-neutral measure, expectation of the square of volatility is reduced to a continuous-time deterministic delay differential equation. (**5**) The by-product of this exhibits the hereditary effects on the mean-square volatility process. (**6**) Using a numerical scheme, a time-series model is developed and utilized to estimate the state and parameters of the dynamic model. In fact, the developed time-series model includes the extended GARCH model as special case. (**7**) Using the Henry Hub natural gas data set, the usefulness of the linear interconnected stochastic models is outlined.

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Chapter 1

Preliminary Concepts and Tools

1.1 Introduction

In this chapter, we shall provide a number of basic definitions and important results which shall be used in later chapters.

1.2 General Notations

$i.e.$	:	that is.
a.s	:	almost surely.
$G := H$	:	G is defined by H or G is denoted by H.
$G(x) \equiv H(x)$	:	$G(x)$ and $H(x)$ are identically equal.
$\emptyset$	:	the empty set.
G^T	:	the transpose of G.
$a \vee b$	:	the maximum of a and b.
$f : A \to B$	:	the mapping f from A to B.
$\mathbb{Z}$	:	set of integers
$I_a(b) = I(a,b)$	:	the set $\{x \in \mathbb{Z} :\ a \leq x \leq b\}$.
$\mathbb{R}$	:	the real line.
$\mathbb{R}^n$	:	the n-dimensional Euclidean space.
$\mathbb{R}^+$	:	the set of all nonnegative real numbers $[0, \infty)$.
$\mathbb{R}_+$	:	the set of all positive real numbers $(0, \infty)$.
$\mathbb{R}^{n\times m}$	:	the space of real $n\times$ m-matrices.
$\mathcal{C}$	:	the family of all real-valued continuous functions.

$\mathcal{C}_n$: the family of all real-valued functions $V(x)$ which are continuously n-times differentiable in x.

$\mathcal{C}_{n,m}$: the family of all real-valued functions $V(t,x)$ which are continuously n-times differentiable in t and m-times differentiable in x.

$a \neq b$: a is not equal to b.

$a \in A$: a is an element of A.

$\|A\| = \|A\|_2$: the Euclidean norm of A.

$trA = traceA$: the trace of a square matrix A.

$det\ A$: determinant of square matrix A.

V_x : $\nabla V = (V_{x_1}, ..., V_{x_n}) = \left(\frac{\partial V}{\partial x_1}, ..., \frac{\partial V}{\partial x_n}\right)$.

V_{xx} : $\left(V_{x_i x_j}\right)_{n\times n} = \left(\frac{\partial^2 V}{\partial x_i x_j}\right)_{n\times n}$

1.3 Stochastic Differential Equation

Given a n-dimensional stochastic process on $t \geq t_0$, a typical It$\hat{o}$-Doob type stochastic differential equation is given by

$$d\mathbf{x} = \mu(t,\mathbf{x})dt + \sigma(t,\mathbf{x})d\mathbf{W}(t), \ \ \mathbf{x}(t_0) = \mathbf{x}_0, \tag{1.1}$$

where $\mu : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$; $\sigma : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ and $W(t) = (W_1(t), ..., W_m(t))^T$ is a standard Wiener process on a filtered probability space $(\Omega, \mathcal{F}_t, (\mathcal{F}_t)_{t\geq 0}, \mathcal{P})$; the filtration function $(\mathcal{F})_{t\geq 0}$ is right-continuous, and each $\mathcal{F}_t$ with $t \geq 0$ contains all $\mathcal{P}$-null sets in $\mathcal{F}_t$. We say μ is the drift coefficient while σ is the diffusion coefficient.

Next, we state the It$\hat{o}$-Doob Lemma.

THEOREM 1.1 *Let u be a continuous map from $\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ such that $u(t,\boldsymbol{x})$ has continuous partial derivatives up to second order in $\boldsymbol{x}$ and to first order in t, then the process $u(t,\boldsymbol{x}(t))$ also has an Itô-Doob differential, and*

$$du(t,\boldsymbol{x}) = \boldsymbol{L}u(t,\boldsymbol{x})dt + u_{\boldsymbol{x}}\sigma(t,\boldsymbol{x})d\boldsymbol{W}(t) \tag{1.2}$$

where $\boldsymbol{L}$ is a differential operator defined by:

$$\boldsymbol{L}u(t,\boldsymbol{x}) = u_t(t,\boldsymbol{x}) + u_{\boldsymbol{x}}(t,\boldsymbol{x})\mu(t,\boldsymbol{x}) + \frac{1}{2}trace\left(\sigma^T(t,u_t(t,\boldsymbol{x}))u_{\boldsymbol{xx}}(t,\boldsymbol{x})\sigma(t,u_t(t,\boldsymbol{x}))\right). \tag{1.3}$$

The following theorem concerns the classical existence, uniqueness and certain other properties of the solution of (1.1).

THEOREM 1.2 *[79] Assume that there exist two positive constants K and $\bar{K}$ such that*

- *(Lipschitz condition): for all $\boldsymbol{x}$, $\boldsymbol{y} \in \mathbb{R}^n$ and $t \in [t_0, T]$*

$$\|\mu(t,\boldsymbol{x}) - \mu(t,\boldsymbol{y})\|^2 \vee \|\sigma(t,\boldsymbol{x}) - \sigma(t,\boldsymbol{y})\|^2 \quad \leq \quad K\|x-y\|^2;$$

- *(Linear growth condition): for all $(t,\boldsymbol{x}) \in [t_0, T] \times \mathbb{R}^n$*

$$\|\mu(t,\boldsymbol{x})\|^2 \vee \|\sigma(t,\boldsymbol{x})\|^2 \quad \leq \quad \bar{K}\left(1 + \|\boldsymbol{x}\|^2\right),$$

where $\vee$ is the $\max$ symbol. Then there exists a unique solution $\boldsymbol{x}(t)$ to (1.1).

In the following, we state a result that exhibits the existence of non-linear stochastic differential equations.

THEOREM 1.3 *[[57], Thm 3.5] Suppose that the local solution of (1.1) exists on every cylinder $[t_0, \infty) \times U_n$, where $U_n = \{\boldsymbol{x} \in \mathbb{R}^n \ : \ \|\boldsymbol{x}\| < n\}$. Moreover, suppose that there exists a nonnegative function $V \in \mathcal{C}_{1,2}$ such that for some constant $c > 0$*

$$\begin{cases} \boldsymbol{L}V \leq cV \\ V_n = \inf\limits_{\|\boldsymbol{x}\|>n} V(t,\boldsymbol{x}) \to \infty \;\; as \;\; n \to \infty, \end{cases} \tag{1.4}$$

where the $\boldsymbol{L}$-operator is defined in (1.3). Then, for every random variable $\boldsymbol{x}(t_0)$ independent of the process $W_i(t) - W_i(t_0)$, there exists a solution $\boldsymbol{x}(t)$ of the system of stochastic differential equation (1.1) which is almost surely continuous stochastic process and is unique up to equivalence.

1.4 Behavior of Delayed Process

Consider a nonlinear delayed integro-differential equation of the form

$$\frac{dv(t)}{dt} = cv(t) + \beta \int_{-\tau}^{0} v(t+s)ds. \tag{1.5}$$

In order to find approximate solution representation, we need to investigate the behavior of (1.5). For this purpose, we present a result regarding its solution process. Our result is based on results of [64] and [62].

DEFINITION 1.4.1 *A non-constant solution $v(t)$ of* (1.5) *is said to be*

- ***oscillatory*** *if $v(t)$ has arbitrary large number of zeros on $\mathbb{R}^+ = [0, \infty)$, that is, there exists an unbounded sequence $\{t_n \in \mathbb{R}^+\}$ such that $v(t_n) = 0$.*
- ***non-oscillatory*** *if $v(t)$ is not oscillatory, that is, there exist a positive number T such that $v(t)$ is either positive or negative for all $t \geq T$.*

Following the definition in [91],

DEFINITION 1.4.2 *The stochastic integral with respect to Brownian motion $W(t)_{t\in\mathbb{R}^+}$ of any simple predictable process $u : \mathbb{R}^+ \times \Omega \to \mathbb{R}$ of the form*

$$u(t, w) = \sum_{i=1}^{n} F_i \mathbb{1}_{(t_{i-1},t_i]}(t), \quad t \in \mathbb{R}^+, \tag{1.6}$$

is defined by

$$\int_0^\infty u(t)dW(t) = \sum_{i=1}^{n} F_i(W(t_i) - W(t_{i-1})), \tag{1.7}$$

where F_i is an $\mathcal{F}_{t_{i-1}}$ measurable random variable for $i = 1, ..., n$, $u(t) \equiv u(t, w)$.

In the following, we state a result that exhibits the existence of solution of system of non linear equations. For the sake of easy reference, we shall state the Implicit function theorem without proof.

THEOREM 1.4 ***Implicit Function Theorem****[2] Let $\boldsymbol{F} = \{F_1, F_2, ..., F_q\}$ be a vector-valued function defined on an open set $S \in \mathbb{R}^{q+k}$ with values in $\mathbb{R}^q$. Suppose $\boldsymbol{F} \in \mathcal{C}_1$ on S. Let $(\boldsymbol{u}_0; \boldsymbol{v}_0)$ be a point in S for which $\boldsymbol{F}(\boldsymbol{u}_0; \boldsymbol{v}_0) = 0$ and for which the $q \times q$ determinant det $[D_j\boldsymbol{F}_i(\boldsymbol{u}_0; \boldsymbol{v}_0)] \neq 0$. Then there exists a $k-$ dimensional open set $\boldsymbol{T}_0$ containing $\boldsymbol{v}_0$ and unique vector-valued function $\boldsymbol{g}$, defined on $\boldsymbol{T}_0$ and having values in $\mathbb{R}^q$, such that $\boldsymbol{g} \in \mathcal{C}_1$ on $\boldsymbol{T}_0$, $\boldsymbol{g}(\boldsymbol{v}_0) = \boldsymbol{u}_0$, and $\boldsymbol{F}(\boldsymbol{g}(\boldsymbol{v}); \boldsymbol{v}) = 0$ for every $\boldsymbol{v} \in \boldsymbol{T}_0$.*

Chapter 2

Linear Stochastic Modeling of Energy Commodity Spot Price Processes with Delay in Volatility

2.1 Introduction

In real world situations, the expected spot price of energy commodities and its measure of variation are not constant. This is because of the fact that a spot price is subject to random environmental perturbation. Moreover, some statistical studies of stock price [8] raised the issue of market's delayed response. This indeed causes the price to drift significantly away from the market quoted price. It is well recognized that time-delay models in economics [41, 56, 123] are more realistic than the models without time-delay. Discrete-time stochastic volatility models [9, 38] have been developed in economics. Recently, a survey paper by Hansen and Lunde [46] has estimated these types of models and concluded that the performance of the GARCH(1,1) is better than any other model. Furthermore, Cox-Ingersoll-Ross(CIR) developed a mean reverting interest rate model that was based on the mean-level interest rate as exponentially weighted integral of past history of interest rate and the relationship between level dependent volatility and the square root of the interest rate [19]. Employing the Ornstein Uhlenbeck [126] and Cox-Ingersoll-Ross(CIR) [19] processes, Heston developed a stochastic model for the volatility of stock spot asset. Recently [51], a continuous time stochastic volatility models have been generalized.

In this work, using basic economic principles, we systematically develop both deterministic and stochastic dynamic models for the log-spot price process. In addition, by treating a diffusion coefficient parameter in the non-seasonal log-spot price dynamic system as a stochastic volatility function of log-spot price, a stochastic model for interconnected system of log-spot price, expected log-spot price and hereditary volatility process is developed. Introducing a numerical scheme, a time-series model is developed and it is utilized to estimate the system parameters. The organization of this study is as follows:

In Section 2.2, we develop a stochastic interconnected models for energy commodity spot price and give an illustration by analyzing Henry Hub Natural gas daily Spot price from 1997 to 2011. In

Section 2.3, we obtain closed form solutions of the log of spot and the expected log of spot prices. In Section 2.4, by outlining the risk-neutral dynamics of price process, sufficient conditions are given to ensure that the risk-neutral dynamics of our model is equivalent to the developed model in Section 2.2. Furthermore, it is shown that the mean of the square of volatility under the risk-neutral measure is a deterministic continuous-time delay differential equation. In addition, sufficient conditions are also given to investigate both the oscillatory and non-oscillatory behavior of the expected value of square of volatility [62, 64].

2.2 Model Derivation

We denote $S(t)$ to be the spot price for a given energy commodity at a time t. Since the price of energy commodity are non-negative, to minimize ambiguity and for the sake of simplicity, it is expressed as an exponential function of the following form;

$$S(t) = \exp\left(x_2(t) + f(t)\right), \tag{2.1}$$

where $x_2(t)$ stands for the nonseasonal log of the spot price at time t, $f(t)$ is the price at t influenced by the seasonality and it is considered as a Fourier series comprising of linear combinations of sine and cosine functions;

$$f(t) = A_0 + \sum_{k=1}^{N}\left(A_k \cos\left[\frac{2\pi kt}{P}\right] + B_k \sin\left[\frac{2\pi kt}{P}\right]\right), \tag{2.2}$$

where $P, A_0, A_k, B_k, k = 1...N$ are all constant parameters. P is the period which represents the number of trading days in a year. Without loss in generality, we choose $N = 2$. By modeling the seasonal term this way, we are able to account for the peak season high price and off peak season low price of gas.

We present the dynamics for the spot price process.

2.2.1 Deterministic Non-Seasonal Log-Spot Price Dynamic Model

Under the basic economic principle of demand and supply processes, the price of a energy commodity will remain within a given finite upper bound. Let $\kappa > 0$ be the expected upper limit of $x_2(t)$.

In a real world situation, the nonseasonal log of spot price is governed by the spot price dynamic process. This leads to a development of dynamic model for the nonseasonal process $x_2(t)$. In this case, $\kappa - x_2(t)$ characterizes the market potential for $x_2(t)$ per unit of time at a time t. This market potential is influenced by the underlying market forces on the nonseasonal $\log$ of spot price, $x_2(t)$. This leads to the following principle regarding the dynamic of non-seasonal log-spot price process of energy goods. The change in nonseasonal log of spot price of the energy commodity $\Delta x_2(t) = x_2(t+\Delta t) - x_2(t)$ over the interval of length $|\Delta t|$ is directly proportional to the market potential price.

$$\Delta x_2(t) \propto (\kappa - x_2(t))\Delta t. \tag{2.3}$$

This implies

$$dx_2(t) = \gamma(\kappa - x_2(t))dt, \tag{2.4}$$

where γ is a positive constant of proportionality, $dx_2(t)$ and dt are differentials of $x_2(t)$ and t respectively. From this mathematical model, we note that as the nonseasonal log price, $x_2(t)$ fall below the expected price κ, $\kappa - x_2(t)$ is positive. Hence $x_2(t)$ is increasing at the constant rate γ per unit size of $\kappa - x_2(t)$ per unit time. On the other hand, if the nonseasonal log price $x_2(t)$ is above the expected price κ, then $\kappa - x_2(t)$ is negative and hence $x_2(t)$ decreases at the rate γ per unit size per unit time.

From (2.3), we note that the steady-state or equilibrium state nonseasonal log of spot price is given by

$$x_2^* = \kappa. \tag{2.5}$$

In the real world situation, the expected price of the nonseasonal log spot price κ is not a constant parameter. Therefore, we consider the expected nonseasonal log of spot price to be the mean of nonseasonal log spot price, $x_2(t)$, at time t denoted by $x_1(t)$. Under this assumption, (2.4) reduces to

$$dx_2(t) = \gamma(x_1(t) - x_2(t))dt. \tag{2.6}$$

Moreover, in order to preserve the equilibrium of nonseasonal log spot price ($\kappa = x_2^*$), we further assume that the mean of nonseasonal spot price process is operated under the principle described by (2.3).

$$\Delta x_1(t) \propto (\kappa - x_1(t))\Delta t \tag{2.7}$$

and hence

$$dx_1(t) = \mu(\kappa - x_1(t))dt, \tag{2.8}$$

where μ is a positive constant of proportionality. From (2.6) and (2.7), the mathematical model for the deterministic nonseasonal spot price process is described by the following system of differential equations:

$$\begin{cases} dx_1(t) = \mu(\kappa - x_1(t))dt, \\ dx_2(t) = \gamma(x_1(t) - x_2(t))dt. \end{cases} \tag{2.9}$$

2.2.2 Stochastic Non-Seasonal Log-Spot Price Dynamic Model

We note that in (2.3), κ is not just the time-varying deterministic log of spot price, instead it is a stochastic process describing random environmental perturbations as follows:

$$\kappa = x_1(t) + e_2(t) \tag{2.10}$$

where $x_1(t)$ is the deterministic part and $e_2(t)$ is the stochastic white noise process. From this, (2.4) becomes

$$\begin{aligned} dx_2(t) &= \gamma(x_1(t) + e_2(t) - x_2(t))dt \\ &= \gamma(x_1(t) - x_2(t))dt + \gamma e_2(t)dt \\ &= \gamma(x_1(t) - x_2(t))dt + \sigma(t, x_2(t))dW_2(t). \end{aligned} \tag{2.11}$$

where $\sigma(t, x_2(t))dW_2(t) = \gamma e_2(t)dt$ and $dW_2(t) \sim \mathcal{N}(0, dt)$.

Following the argument used in the derivation of (2.11), the dynamic from (2.7) reduces to

$$dx_1(t) = \mu(\kappa - x_1(t))dt + \delta dW_1(t) \tag{2.12}$$

where $\delta > 0$ is a constant and $dW_1(t) \sim \mathcal{N}(0, dt)$.

From (2.12) and (2.11), the mathematical model for the stochastic nonseasonal spot price process is described by the following system of differential equations:

$$\begin{cases} dx_1 = \mu(\kappa - x_1)dt + \delta dW_1(t), \\ dx_2 = \gamma(x_1 - x_2)dt + \sigma(t, x_2)dW_2(t). \end{cases} \tag{2.13}$$

2.2.3 Continuous Stochastic Volatility Model with Delay

When considering energy commodities, the measure of variation of the spot price under random environmental perturbation is not predictable, because it depends on nonseasonal log of spot price. Bernard and Thomas [8] in their work raised the issue of market's delayed response. They observed changes in drift returns that leads to two possible explanations. First explanation suggests that a part of the price response to new information is delayed. The second explanation suggests that researchers fail to adjust fully a raw return for risks, because the capital-asset-pricing model used to calculate the abnormal return is either incomplete or incorrect estimation. In this study, we incorporate the past history of nonseasonal log of spot price in the coefficient of diffusion parameter, that is, the volatility $\sigma(t, x_2(t))$ of the spot price follows the GARCH model [140]. It is assumed that the measure of variation of random environmental perturbations of $x_1(t)$ is constant. Under these assumptions, we propose an interconnected mean-reverting non-seasonal stochastic model for mean log-spot price, log-spot price,and volatility as follows:

$$\begin{aligned} dx_1 &= \mu(\kappa - x_1)dt + \delta dW_1(t), \quad x_1(t_0) = x_{01} \\ dx_2 &= \gamma(x_1 - x_2)dt + \sigma(t, x_2)dW_2(t), \quad x_2(t_0) = x_{02} \\ d\sigma^2(t, x_2) &= \Big[\alpha + \beta \left[\int_{t-\tau}^{t} \sigma(s, x_2)e^{-\gamma(t-s)}dW_2(s) + \delta \int_{t-\tau}^{t} \phi(s,t)dW_1(s)\right]^2 \\ &\quad + c\sigma^2(t, x_2)\Big]\, dt, \end{aligned} \tag{2.14}$$

where

$$\phi(a, b) = \frac{\gamma}{\mu-\gamma}\left(e^{-\gamma(b-a)} - e^{-\mu(b-a)}\right), \ \gamma, \mu \text{ are defined in } (2.4)\ and\ (2.8),\ a, b \in \mathbb{R}. \tag{2.15}$$

For the sake of completeness, we assume the following

$\mathbf{H}_1$: $x_{2t}(\theta) = x_2(t + \theta)$, $\theta \in [-\tau, 0]$, γ, μ, $\delta \in \mathbb{R}_+$, $\alpha, \beta,\ c$ are in $\mathbb{R}$, (we will later show that $-2 < c < 0$), $\sigma : [0, T] \times \mathcal{C} \to \mathbb{R}$ is a continuous mapping, $\mathcal{C}$ is the Banach space of continuous functions defined on $[-\tau, 0]$ into $\mathbb{R}$ and equipped with the supremum norm; $W_1(t)$ and $W_2(t)$ are standard Wiener process defined on a filtered probability space $(\Omega, \mathcal{F}_t, (\mathcal{F}_t)_{t\geq 0}, \mathcal{P})$, the filtration function $(\mathcal{F})_{t\geq 0}$ is right-continuous, and for $t \geq 0$, $\mathcal{F}_t$ contains all $\mathcal{P}$-null sets. We know that system (2.14) can be re-written as

$$d\mathbf{x} = [\mathbf{A}\ \mathbf{x} + \mathbf{p}]\, dt + \sum(t, x_2)\ d\mathbf{W}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0, \tag{2.16}$$

where

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},\ \mathbf{A} = \begin{bmatrix} -\mu & 0 \\ \gamma & -\gamma \end{bmatrix},\ \mathbf{p}(t) = \begin{bmatrix} \mu\kappa \\ 0 \end{bmatrix},\ \sum(t, x_2(t)) = \begin{bmatrix} \delta & 0 \\ 0 & \sigma(t, x_2(t)) \end{bmatrix},$$

$$\mathbf{W}(t) = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix},\ \mathbf{x}_0 = \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix}.$$

Moreover, (2.16) can be considered as a system of nonlinear It$\hat{o}$-Doob type stochastic perturbed system of the following deterministic linear system of differential equations

$$d\mathbf{x} = \mathbf{A}\mathbf{x}dt. \tag{2.17}$$

In the following, we present an illustration to justify the structure of log spot price dynamic model.

2.2.4 Illustration

We present an illustrate the above described interconnected stochastic dynamic model for non-seasonal log spot price of energy commodity under the influence of random perturbations on mean-level and delayed volatility.

We consider the Henry Hub Natural Gas Daily spot price from 1997 to 2011.

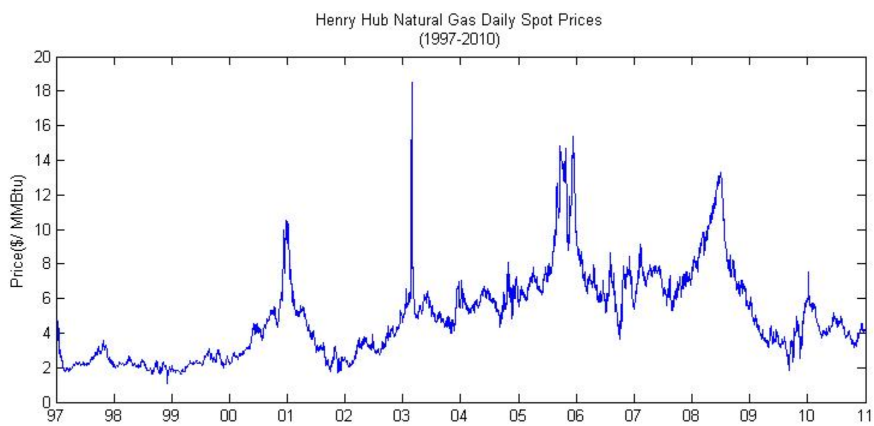


Figure 1.: Plot of Henry Hub Daily Natural Gas Spot Prices, 1997-2011

We can clearly see that

- Prices appear as being randomly driven and clearly non-negative
- There is a tendency of spot prices to move back to their long term level (mean reversion).
- There are sudden large changes in spot prices (jumps/spikes).
- There is an unpredictability of spot price volatility.

A summary of the statistics is presented in Table 1 below. We find that $\ln\left[\frac{S(t+1)}{S(t)}\right]$ has the smallest variance. Thus, it suggests a good candidate for our modeling. Hence, we use the logarithmic price, rather than the raw price data for our model.

Table 1: Descriptive statistics of Henry Hub daily natural gas spot prices, 1997-2010

	Mean	Variance	Skewness	Kurtosis	Minimum	Maximum
S_t	4.9519	2.4966	1.0391	4.3491	1.05	18.48
$S_{t+1} - S_t$	-0.0001142	0.3189	-0.7735	191.8911	-8.01	6.50
$ln(S_t)$	1.4754	0.5048	-0.0465	2.1540	0.0488	2.9167
$ln(\frac{S_{t+1}}{S_t})$	2.8485e-5	0.0473	0.4814	22.0473	-0.56	0.5657

2.3 Closed Form Solution

In this section, we find the solution representation of (2.16) in terms of the solution of unperturbed system of differential deterministic (2.17). This is achieved by employing method of variation of constants parameter [70].

THEOREM 2.1 ***(Closed Form Solution)***
Let $\boldsymbol{x}(t) = \boldsymbol{x}(t, t_0; \boldsymbol{x}_0)$ and $\boldsymbol{y}(t, t_0; \boldsymbol{x}_0) = \Phi(t, t_0)\boldsymbol{x}_0$ be the solutions of the perturbed and unperturbed system of differential equations (2.16) and (2.17) respectively. Then

$$\begin{aligned} \boldsymbol{x}(t) &= \begin{bmatrix} e^{-\mu(t-t_0)} & 0 \\ \phi(t_0, t) & e^{-\gamma(t-t_0)} \end{bmatrix} \boldsymbol{x}_0 + \begin{bmatrix} \kappa\left(1 - e^{-\mu(t-t_0)}\right) \\ \omega(t_0, t) \end{bmatrix} \\ & + \int_{t_0}^{t} \begin{bmatrix} \delta e^{-\mu(t-s)} dW_1(s) \\ \delta\phi(s, t) dW_1(s) + \sigma(s, x_2(s)) e^{-\gamma(t-s)} dW_2(s) \end{bmatrix}, \end{aligned} \tag{2.18}$$

where

$$\boldsymbol{x}_0 = \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix}, \tag{2.19}$$

$$\omega(a, b) = \kappa\left(\left[1 - e^{-\gamma(b-a)}\right] - \phi(a, b)\right), \quad a, b \in \mathbb{R}. \tag{2.20}$$

and ϕ is defined in (2.15); the fundamental solution, $\Phi(t)$ of (2.17) is given by

$$\Phi(t, t_0) = \begin{bmatrix} e^{-\mu(t-t_0)} & 0 \\ \phi(t_0, t) & e^{-\gamma(t-t_0)} \end{bmatrix} \tag{2.21}$$

Proof. The result follows by imitating the eigenvalue type method described in [69, 70]. Therefore

$$x_1(t) = e^{-\mu(t-t_0)} x_{01} + \kappa\left(1 - e^{-\mu(t-t_0)}\right) + \delta \int_{t_0}^{t} e^{-\mu(t-s)} dW_1(s), \tag{2.22}$$

$$x_2(t) = \phi(t_0,t)x_{01} + e^{-\gamma(t-t_0)}x_{02} + \omega(t_0,t) + \delta\int_{t_0}^{t}\phi(s,t)dW_1(s) + \int_{t_0}^{t}\sigma(s,x_2(s))e^{-\gamma(t-s)}dW_2(s). \tag{2.23}$$

$\square$

In the following, we present the statistical properties of the solutions (2.22) and (2.23).

THEOREM 2.2 *Under the hypothesis of Theorem* 2.1*, we have*

$$\begin{aligned}
\mathbb{E}[\boldsymbol{x}(t)] &= \begin{bmatrix} e^{-\mu(t-t_0)} & 0 \\ \phi(t_0,t) & e^{-\gamma(t-t_0)} \end{bmatrix}\boldsymbol{x}_0 + \begin{bmatrix} \kappa\left(1-e^{-\mu(t-t_0)}\right) \\ \omega(t_0,t) \end{bmatrix}, \\
\mathit{var}[\boldsymbol{x}(t)] &= \begin{bmatrix} \delta^2\int_{t_0}^{t} e^{-2\mu(t-s)}ds \\ \int_{t_0}^{t}\mathbb{E}(\sigma^2(s,x_2(s)))e^{-2\gamma(t-s)}ds + \delta^2\int_{t_0}^{t}\phi^2(s,t)ds \end{bmatrix}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\lim_{t\to\infty}\mathbb{E}[\boldsymbol{x}(t)] &= \begin{bmatrix} \kappa \\ \kappa \end{bmatrix}, \\
\lim_{t\to\infty}\mathit{var}[\boldsymbol{x}(t)] &= \begin{bmatrix} \frac{\delta^2}{2\mu} \\ \lim_{t\to\infty}\frac{\mathbb{E}(\sigma^2(t,x_2(t)))}{2\gamma} + \frac{\delta^2}{2\mu}\left[\frac{\gamma}{\mu+\gamma}\right] \end{bmatrix}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\lim_{t\to\infty}\mathbb{E}[x_1(t)] &= \lim_{t\to\infty}\mathbb{E}[x_2(t)] = \kappa, \\
\lim_{t\to\infty}\mathit{var}(x_1(t)) &= \frac{\delta^2}{2\mu}
\end{aligned}$$

Proof. From (2.18), we observe that

$$\mathbb{E}[\mathbf{x}(t)] = \begin{bmatrix} e^{-\mu(t-t_0)} & 0 \\ \phi(t_0,t) & e^{-\gamma(t-t_0)} \end{bmatrix}\mathbf{x}_0 + \begin{bmatrix} \kappa\left(1-e^{-\mu(t-t_0)}\right) \\ \omega(t_0,t) \end{bmatrix}.$$

Hence,

$$\begin{aligned}
\mathbb{E}\left(x_1(t)\right) &= x_{01}e^{-\mu(t-t_0)} + \kappa\left(1-e^{-\mu(t-t_0)}\right) \\
\mathbb{E}\left(x_2(t)\right) &= x_{02}e^{-\gamma(t-t_0)} + \phi(t_0,t)x_{01} + \omega(t_0,t) \\
\mathit{var}\left(x_1(t)\right) &= \delta^2\int_{t_0}^{t}e^{-2\mu(t-s)}ds = \frac{\delta^2}{2\mu}\left[1-e^{-2\mu(t-t_0)}\right] \\
\mathit{var}\left(x_2(t)\right) &= \int_{t_0}^{t}\mathbb{E}(\sigma^2(s,x_2(s)))e^{-2\gamma(t-s)}ds + \delta^2\int_{t_0}^{t}\phi^2(s,t)ds.
\end{aligned}$$

The result follows by taking the limits as $t\to\infty$. $\square$

REMARK 1 From Theorem 2.2, we observe that on the long-run, the mean-level of $x_2(t)$ and $x_1(t)$ are the same and it is given by κ.

2.4 Risk-Neutral Dynamics and Pricing

In order to minimize the risk of usage of mathematical model (2.16), we incorporate the risk neutral measure. From the dynamic nature of (2.16), it is known [20] that this model has affine multi-factor structure. In the following, we present a risk neutral measure induced by this type of model. This indeed leads to a risk neutral dynamic model with respect to (2.16). Christa Cuchiero, [20], showed in their work that the market price of risk $\Theta(t) = (\Theta_1(t), ..., \Theta_n(t))$ with respect to the stochastic differential equation (1.1) is given by

$$\Theta_i(t) = \frac{\frac{\eta(t,\mathbf{x}_t)}{P(t,T)} - r(t)}{\frac{n\zeta_i(t,\mathbf{x}_t)}{P(t,T)}}, \;\; i = 1, 2, ..., n, \tag{2.24}$$

where $P(t, T) = G(t, \mathbf{x})$ is the zero-coupon bond price, $r(t)$ is the short-term rate factor for the risk-free borrowing or lending at time t over the interval $[t, t + dt]$, and $\eta(t, \mathbf{x}_t)$, $\zeta(t, \mathbf{x})$ are defined by

$$\begin{aligned} \eta(t, \mathbf{x}) &= \mathbf{L}G(t, \mathbf{x}), \\ \zeta(t, \mathbf{x}) &= \frac{\partial G(t,\mathbf{x})}{\partial \mathbf{x}}\sigma(t, \mathbf{x}), \end{aligned} \tag{2.25}$$

where $\frac{\partial G(t,\mathbf{x})}{\partial \mathbf{x}}$ is the gradient of G, and the $\mathbf{L}$-operator is defined in (1.3)..

In fact, since our price model $\mathbf{X}(t) = (x_1(t), x_2(t))^T$ (2.16) is an affine multi-factor model, the short-term rate factor $r(t)$ and the zero-coupon bond price $P(t, T)$ can be represented as

$$\begin{aligned} r(t) &= g + \mathbf{h}\mathbf{X}(t) \\ P(t, T) &= \exp\left(a(t, T) + \mathbf{B}(t, T)\mathbf{X}(t)\right), \end{aligned} \tag{2.26}$$

where $g \in \mathbb{R}$, $\mathbf{h} \in \mathbb{R}^2$, $a(t, T)$ and $\mathbf{B}(t, T) = (B_1(t, T), B_2(t, T), ..., B_n(t, T))$ are arbitrary smooth functions. For $n = 2$, from (2.24) and (2.25), the market price of risk $\Theta(t) = (\theta_1(t), \theta_2(t))$ is given by

$$\Theta(t) = \mathbf{a} + \mathbf{b}(t)\mathbf{X}(t), \tag{2.27}$$

where

$$
\begin{aligned}
\mathbf{a} &= \begin{bmatrix} a_{1,0} \\ a_{2,0} \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}, \ \mathbf{X}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \\
a_{2,0}(t) &= \frac{\frac{1}{2}B_2^2(t,T)\sigma^2(t,\mathbf{x}) + \frac{da(t,T)}{dt} - g}{B_2(t,T)\sigma(t,\mathbf{x})}, \\
a_{2,1}(t) &= \frac{\gamma B_2(t,T) - h_1}{B_2(t,T)\sigma(t,\mathbf{x})}, \\
a_{2,2}(t) &= \frac{-\gamma B_2(t,T) - h_2}{B_2(t,T)\sigma(t,\mathbf{x})}, \\
a_{1,0}(t) &= \frac{\frac{1}{2}B_1^2(t,T)\delta^2 + \mu\kappa B_1(t,T) + \frac{da(t,T)}{dt} - g}{B_1(t,T)\delta}, \\
a_{1,1}(t) &= \frac{-\mu B_1(t,T) - h_1}{B_1(t,T)\delta}, \\
a_{1,2}(t) &= 0.
\end{aligned}
$$

We incorporate a market price of risk process that gives a risk-neutral dynamics of the same class as (2.16) in the following lemma.

LEMMA 2.1 *Let us assume that a and $B_i,\ i = 1,2$ in (2.26) are arbitrary constants. The market price of risk processes reduces to;*

$$
\begin{aligned}
\theta_1(t) &= a_{1,0} + a_{1,1}x_1(t) + a_{1,2}x_2(t) && (2.28) \\
\theta_2(t) &= a_{2,0}(t) + a_{2,1}(t)x_1(t) + a_{2,2}(t)x_2(t). && (2.29)
\end{aligned}
$$

In addition, let us assume that $\theta_i,\ i = 1,2$ satisfy the Novikov's condition [108] with the $\bar{\mathcal{P}}$-Wiener process;

$$
\begin{aligned}
\bar{W}_1(t) &= W_1(t) + \int_{t_0}^{t}\theta_1(u)du \\
\bar{W}_2(t) &= W_2(t) + \int_{t_0}^{t}\theta_2(u)du.
\end{aligned} \tag{2.30}
$$

and

$$
\boldsymbol{C}_1 : \begin{cases} h_1 + h_2 = 0, \\ a_{2,0}(t) = a_{1,2} = 0, \\ \bar{\gamma} = \frac{h_1}{B_2}, \ \ \bar{\mu} = \frac{h_2}{B_1}, \\ \bar{\mu}\bar{\kappa} = \mu\kappa - \delta a_{1,0}, \end{cases} \tag{2.31}
$$

where $h_1,\ h_2$ are arbitrary real numbers; $\mu,\ \kappa$ and δ are defined in (2.14), $\theta_i,\ a_{i,j},\ i = 1,2$, $j = 0,1,2$ are defined in (2.27).

Then the risk-neutral dynamics of $x_1(t)$ and $x_2(t)$ remain within the same class,

$$d\boldsymbol{x} = \left[\bar{\boldsymbol{A}}\,\boldsymbol{x} + \bar{\boldsymbol{p}}\right]dt + \sum(t, x_2)\ \ d\bar{\boldsymbol{W}}(t), \quad \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \tag{2.32}$$

satisfying $\boldsymbol{H}_1$,where

$$\boldsymbol{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},\ \bar{\boldsymbol{A}} = \begin{bmatrix} -\bar{\mu} & 0 \\ \bar{\gamma} & -\bar{\gamma} \end{bmatrix},\ \bar{\boldsymbol{p}}(t) = \begin{bmatrix} \bar{\mu}\bar{\kappa} \\ 0 \end{bmatrix},\ \sum(t, x_2(t)) = \begin{bmatrix} \delta & 0 \\ 0 & \sigma(t, x_2(t)) \end{bmatrix},$$

$$\bar{\boldsymbol{W}}(t) = \begin{bmatrix} \bar{W}_1 \\ \bar{W}_2 \end{bmatrix},\ \boldsymbol{x}_0 = \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix}.$$

Moreover, it satisfies Hypothesis $\boldsymbol{H}_1$. Hence,

$$\begin{aligned} dx_1 &= \bar{\mu}(\bar{\kappa} - x_1)dt + \delta d\bar{W}_1(t), \\ dx_2 &= \bar{\gamma}\left(x_1 - x_2\right)dt + \sigma(t, x_2)d\bar{W}_2(t). \end{aligned} \tag{2.33}$$

Proof. The proof follows by substituting (2.30) and $\mathbf{C}_1$ into (2.16). □

REMARK 2 Under the assumption of Lemma 2.1, it is obvious that the solution to (2.32) is given by

$$\begin{aligned} \mathbf{x}(t) &= \begin{bmatrix} e^{-\bar{\mu}(t-t_0)} & 0 \\ \bar{\phi}(t_0, t) & e^{-\bar{\gamma}(t-t_0)} \end{bmatrix}\mathbf{x}_0 + \begin{bmatrix} \bar{\kappa}\left(1 - e^{-\bar{\mu}(t-t_0)}\right) \\ \bar{\omega}(t_0, t) \end{bmatrix} \\ &\quad + \int_{t_0}^{t} \begin{bmatrix} \delta e^{-\bar{\mu}(t-s)}dW_1(s) \\ \delta\bar{\phi}(s,t)dW_1(s) + \sigma(s, x_2(s))e^{-\bar{\gamma}(t-s)}dW_2(s) \end{bmatrix}, \\ \mathbf{x}(t_0) &= \mathbf{x}_0, \end{aligned} \tag{2.34}$$

where $\bar{\phi}$ and $\bar{\omega}$ are defined as

$$\begin{aligned} &\bar{\phi}(a,b) = \tfrac{\bar{\gamma}}{\bar{\mu}-\bar{\gamma}}\left(e^{-\bar{\gamma}(b-a)} - e^{-\bar{\mu}(b-a)}\right) \\ &\bar{\omega}(a,b) = \bar{\kappa}\left(\left[1 - e^{-\bar{\gamma}(b-a)}\right] - \bar{\phi}(a,b)\right). \end{aligned} \tag{2.35}$$

In the following, we state a result with regards to (2.34).

LEMMA 2.2 *Under the assumption $\boldsymbol{H}_1$, (2.34) is equivalent to*

$$\begin{aligned} \boldsymbol{x}(t) &= \begin{bmatrix} e^{-\bar{\mu}\tau} & 0 \\ \bar{\phi}(0,\tau) & e^{-\bar{\gamma}\tau} \end{bmatrix}\boldsymbol{x}(t-\tau) + \begin{bmatrix} \bar{\kappa}\left(1 - e^{-\bar{\mu}\tau}\right) \\ \bar{\omega}(0,\tau) \end{bmatrix} \\ &\quad + \int_{t-\tau}^{t} \begin{bmatrix} \delta e^{-\bar{\mu}(t-s)}d\bar{W}_1(s) \\ \delta\bar{\phi}(s,t)d\bar{W}_1(s) + \sigma(s, x_2(s))e^{-\bar{\gamma}(t-s)}d\bar{W}_2(s) \end{bmatrix}. \end{aligned} \tag{2.36}$$

where $\boldsymbol{x}(t-\tau) = \begin{bmatrix} x_1(t-\tau) \\ x_2(t-\tau) \end{bmatrix}$.

Proof. The proof follows by changing the initial time t_0 in (2.34) to $t-\tau$. $\square$

Hence,

$$x_1(t) = x_1(t-\tau)e^{-\bar{\mu}\tau} + \bar{\kappa}(1-e^{-\bar{\mu}\tau}) + \delta\int_{t-\tau}^{t} e^{-\bar{\mu}(t-s)}d\bar{W}_1(s), \tag{2.37}$$

$$\begin{aligned} x_2(t) \quad &= \quad x_2(t-\tau)e^{-\bar{\gamma}\tau} + \bar{\phi}(0,\tau)x_1(t-\tau) + \bar{\omega}(0,\tau) + \textstyle\int_{t-\tau}^{t}\sigma(s,x_2(s))e^{-\bar{\gamma}(t-s)}d\bar{W}_2(s) + \\ &\quad\ \ \delta\textstyle\int_{t-\tau}^{t}\bar{\phi}(u,t)d\bar{W}_1(u). \end{aligned} \tag{2.38}$$

REMARK 3 From (2.38), we have

$$\begin{aligned} \textstyle\int_{t-\tau}^{t}\sigma(s,x_2(s))e^{-\bar{\gamma}(t-s)}d\bar{W}_2(s) + \delta\int_{t-\tau}^{t}\bar{\phi}(u,t)d\bar{W}_1(u) \quad &= \quad x_2(t) - x_2(t-\tau)e^{-\bar{\gamma}(\tau)} \\ &\quad\ \ -x_1(t-\tau)\bar{\phi}(0,\tau) - \bar{\omega}(0,\tau). \end{aligned} \tag{2.39}$$

The dynamics of volatility process under risk-neutral dynamic system is described by

$$d\sigma^2(t,x_2) \quad = \quad \left[\alpha + \beta\left[\int_{t-\tau}^{t}\sigma(s,x_2)e^{-\gamma(t-s)}dW_2(s) + \delta\int_{t-\tau}^{t}\phi(s,t)dW_1(s)\right]^2 \right. \tag{2.40}$$

$$\left. +c\sigma^2(t,x_2)\right]dt. \tag{2.41}$$

We set

$$u(t) = \mathbb{E}_{\bar{\mathcal{P}}}[\sigma^2(t,x_2(t))]. \tag{2.42}$$

Taking the conditional expectation of both sides under the measure $\bar{\mathcal{P}}$, we obtain the following deterministic delay differential equation

$$\begin{aligned} \frac{du(t)}{dt} \quad &= \quad \alpha + \beta\delta^2\int_{t-\tau}^{t}\bar{\phi}(s,t)^2ds + \beta\int_{t-\tau}^{t}u(s)e^{-2\bar{\gamma}(t-s)}ds + cu(t) \\ &= \quad \alpha + \beta\delta^2 D + \beta\int_{t-\tau}^{t}u(s)e^{-2\bar{\gamma}(t-s)}ds + cu(t) \end{aligned}$$

where

$$\begin{aligned} D \quad &= \quad \int_{t-\tau}^{t}\bar{\phi}(s,t)^2ds \\ &= \quad \left(\frac{\bar{\gamma}}{\bar{\mu}-\bar{\gamma}}\right)^2\left[\frac{1}{2\bar{\gamma}}(1-e^{-2\bar{\gamma}\tau}) - \frac{2}{\bar{\mu}+\bar{\gamma}}(1-e^{(\bar{\mu}+\bar{\gamma})\tau}) + \frac{1}{2\bar{\mu}}(1-e^{-2\bar{\mu}\tau})\right] \end{aligned}$$

Hence

$$\frac{du(t)}{dt} = cu(t) + \beta\int_{t-\tau}^{t}u(s)e^{-2\bar{\gamma}(t-s)}ds + \nu, \tag{2.43}$$

where

$$\nu = \alpha + \beta\delta^2 D.$$

REMARK 4 The equilibrium solution process $u^*(t)$ of (2.43) satisfies the following integral equation

$$cu^*(t) + \beta\int_{t-\tau}^{t} u^*(s)e^{-2\bar{\gamma}(t-s)}ds + \nu = 0, \tag{2.44}$$

since $\frac{du^*(t)}{dt} = 0$. In particular, $u^*(t)$ is as follows;

$$u^*(t) = -\left[\frac{\nu}{c + \frac{\beta}{2\bar{\gamma}}(1 - e^{-2\bar{\gamma}\tau})}\right].$$

Using the transformation

$$v(t) = u(t) - u^*(t) \tag{2.45}$$

we have

$$\begin{aligned}\frac{dv(t)}{dt} &= cv(t) + \beta\int_{t-\tau}^{t} v(s)e^{-2\bar{\gamma}(t-s)}ds + \left[cu^*(t) + \beta\int_{t-\tau}^{t} u^*(s)e^{-2\bar{\gamma}(t-s)}ds + \nu\right]\\ &= cv(t) + \beta\int_{t-\tau}^{t} v(s)e^{-2\bar{\gamma}(t-s)}ds.\end{aligned}$$

Hence,

$$\frac{dv(t)}{dt} = cv(t) + \beta\int_{-\tau}^{0} v(t+s)e^{2\bar{\gamma}s}ds. \tag{2.46}$$

In order to find approximate solution representation, we need to investigate the behavior of (2.46). For this purpose, we present a result regarding its solution process. Our result is based on results of [62] and [64]. Using definition 1.4.1, we prove the following Lemma using the definition of oscillatory and non-oscillatory solution of (2.46).

LEMMA 2.3 *Under the following transformation*

$$v(t) = e^{ct}z(t), \tag{2.47}$$

(2.46) is equivalent to

$$z^{'}(t) = \beta\int_{-\tau}^{0} e^{(c+2\bar{\gamma})s}z(t+s)ds. \tag{2.48}$$

Moreover,

- *(i) for $\beta < 0$ and $\frac{\beta\tau}{c+2\bar{\gamma}}\left[e^{-(c+2\bar{\gamma})\tau} - 1\right] \leq \frac{1}{e}$, every solution of (2.46) is non-oscillatory.*

- *(ii) for* $\beta < 0$ *and* $\frac{\beta\iota}{c+2\bar{\gamma}}\left[e^{-(c+2\bar{\gamma})\tau} - e^{-(c+2\bar{\gamma})\iota}\right] > \frac{1}{e}, \ \ \iota \in (0,\tau)$ *every solution of* (2.46) *oscillates.*

- *(iii) for* $\beta > 0$, (2.46) *has non-oscillatory solutions.*

Proof. To prove (i), suppose that

$$\frac{\beta\tau}{c+2\bar{\gamma}}\left[e^{-(c+2\bar{\gamma})\tau} - 1\right] \le \frac{1}{e}, \quad \beta < 0, \tag{2.49}$$

We observe that every solution of (2.46) is non-oscillatory if and only if every solution (2.48) is non-oscillatory. Therefore, we only need to show that (2.48) has non-oscillatory solution.

Suppose that a solution of (2.48) has the form

$$z(t) = e^{\lambda t} \tag{2.50}$$

where λ is an arbitrary constant which satisfies the equation

$$\lambda = \beta \int_{-\tau}^{0} e^{(c+2\bar{\gamma}+\lambda)s} ds. \tag{2.51}$$

Define

$$G(\lambda) = \lambda - \beta \int_{-\tau}^{0} e^{(c+2\bar{\gamma}+\lambda)s} ds. \tag{2.52}$$

We show that $G(\lambda)$ has at least one real root. From (2.49), (2.50) and nature of β, we note that $G(0) > 0$ and for any $s \in [-\tau, 0]$,

$$\begin{aligned} G(\lambda) &\le \lambda - \beta e^{-\lambda\tau} \int_{-\tau}^{0} e^{(c+2\bar{\gamma})s} ds \\ &= -\beta \int_{-\tau}^{0} e^{(c+2\bar{\gamma})s} ds \left[-e + \exp\left[-e\tau\beta \int_{-\tau}^{0} e^{(c+2\bar{\gamma})s} ds\right]\right] \\ &\le 0 \text{ by } (2.49). \end{aligned}$$

Therefore, (2.51) has at least one real root λ^* that lies between $\beta e \int_{-\tau}^{0} e^{(c+2\bar{\gamma})s} ds$ and 0, showing that (2.46) has non-oscillatory solution. □

Next, we outline the proof for Lemma 5 (ii)

Proof. Suppose

$$\frac{\beta\iota}{c+2\gamma}\left[e^{-(c+2\bar{\gamma})\tau} - e^{-(c+2\bar{\gamma})\iota}\right] > \frac{1}{e}, \ \ \beta < 0, \ \ \text{for any } \iota \in (0,\tau). \tag{2.53}$$

We only need to show that (2.48) oscillates. To verify this, suppose on the contrary that $z(t)$ is a non-oscillatory solution of (2.48). Then for sufficiently large $t_0 > 0$ and without loss in generality,

$z(t) > 0$ for $t \geq t_1$, where $t_1 = t_0 - \tau$. Since $\beta < 0$, $z^{'}(t) < 0$ for $t \geq t_1$. For any $\iota > 0$ such that $-\tau < -\iota < 0$, from (2.46) and (2.48), we have the following inequalities

$$z^{'}(t) \leq \beta \int_{-\tau}^{-\iota} e^{(c+2\bar{\gamma})s} z(t+s)ds, \quad t \geq t_1. \tag{2.54}$$

Hence, for any $s \in (-\tau, -\iota), \ t - \tau < t + s < t - \iota < t$, (2.54) yields

$$z(t) < z(t-\iota) < z(t+s). \tag{2.55}$$

Define

$$w(t) = \frac{z(t-\iota)}{z(t)}, \quad t \geq t_1. \tag{2.56}$$

Note that $w(t) > 1$. Dividing (2.54) by $z(t)$ and using (2.55), we have

$$\frac{z^{'}(t)}{z(t)} - \frac{\beta}{c+2\gamma}\left[e^{-(c+2\bar{\gamma})\iota} - e^{-(c+2\bar{\gamma})\tau}\right] w(t) \quad < \quad \frac{z^{'}(t)}{z(t)} - \beta \int_{-\tau}^{-\iota} e^{(c+2\bar{\gamma})s} \frac{z(t+s)}{z(t)} ds \leq 0.$$

Integrating from $t - \iota$ to t, for $t \geq t_1$,

$$\log z(t) - \log z(t-\iota) - \frac{\beta}{c+2\gamma}\left[e^{-(c+2\bar{\gamma})\iota} - e^{-(c+2\bar{\gamma})\tau}\right] \int_{t-\iota}^{t} w(s)ds \quad \leq \quad 0,$$

and hence

$$\log w(t) \geq \frac{\beta}{c+2\gamma}\left[e^{-(c+2\bar{\gamma})\tau} - e^{-(c+2\bar{\gamma})\iota}\right] \int_{t-\iota}^{t} w(s)ds, \ t \geq t_1 \tag{2.57}$$

Set

$$\lim_{t\to\infty} \inf w(t) = K. \tag{2.58}$$

Since $w(t) > 1$, $K \geq 1$, hence K is either finite or infinite. We show next that none of these cases is true.

Case 1. Assume K is finite. There exist sequence $\{t_n\}$, $t_n \geq t_1 \ \ni \ t_n \to \infty$ and $w(t_n) \to K$ as $n \to \infty$. By integral mean value theorem, $\exists \, c_n \in (t_n - \iota, t_n)$ such that

$$\log w(t_n) \quad \geq \quad \tfrac{\beta\iota}{c+2\gamma}\left[e^{-(c+2\bar{\gamma})\tau} - e^{-(c+2\bar{\gamma})\iota}\right] w(c_n). \tag{2.59}$$

Define $K_1 = \lim_{n\to\infty} w(c_n)$.

Noting that $K_1 \geq K$ and taking limits of (2.59), we have

$$\frac{\log K}{K} \geq \frac{\beta\iota}{c+2\gamma}\left[e^{-(c+2\bar{\gamma})\tau} - e^{-(c+2\bar{\gamma})\iota}\right]. \tag{2.60}$$

Since

$$\max_{K\geq 1} \frac{\log K}{K} = \frac{1}{e}, \tag{2.61}$$

the relation (2.60) implies

$$\frac{\beta\iota}{c+2\gamma}\left[e^{-(c+2\bar{\gamma})\tau}-e^{-(c+2\bar{\gamma})\iota}\right]\leq\frac{1}{e} \tag{2.62}$$

which contradicts (2.53). Hence K is not finite.

Case 2. Assume that K is infinite, from (2.56) and (2.58), we have

$$\lim_{t\to\infty}\left[\frac{z(t-\iota)}{z(t)}\right]=\infty. \tag{2.63}$$

Choose $t_*=t-\alpha,\ \alpha>0$, such that $t-\iota<t_*<t$ for $t\geq t_1$. Integrating both sides of (2.54) from t_* to t and $t-\iota$ to t_*, we have

$$z(t)-z(t_*)-\beta\int_{-\tau}^{-\iota}e^{(c+2\bar{\gamma})s}\left[\int_{t_*}^{t}z(u+s)du\right]ds\leq 0,\ \ t\geq t_1 \tag{2.64}$$

$$z(t_*)-z(t-\iota)-\beta\int_{-\tau}^{-\iota}e^{(c+2\bar{\gamma})s}\left[\int_{t-\iota}^{t_*}z(u+s)du\right]ds\leq 0,\ \ t\geq t_1 \tag{2.65}$$

respectively. We observe that for any $u\in[t_*,t],\ \ s\in[-\tau,-\iota],\ \ u+s<t+s<t-\iota$, hence, $z(t-\iota)<z(t+s)<z(u+s)$, and for any $u\in[t-\iota,t_*]$, $u+s<t_*+s<t_*-\iota$, hence $z(t_*-\iota)<z(t_*+s)<z(u+s)$. Hence (2.64) and (2.65) become

$$z(t)+z(t-\iota)\frac{\beta\alpha}{c+2\gamma}\left[e^{-(c+2\bar{\gamma})\tau}-e^{-(c+2\bar{\gamma})\iota}\right]\leq z(t_*),\ \ t\geq t_1 \tag{2.66}$$

$$z(t_*)+z(t_*-\iota)\frac{\beta(\iota-\alpha)}{c+2\gamma}\left[e^{-(c+2\bar{\gamma})\tau}-e^{-(c+2\bar{\gamma})\iota}\right]\leq z(t-\iota),\ \ t\geq t_1. \tag{2.67}$$

Dividing (2.66) and (2.67) by $z(t)$ and $z(t_*)$ respectively, and using (2.53) and (2.63), we have

$$\lim_{t\to\infty}\frac{z(t_*)}{z(t)}=\lim_{t_*\to\infty}\frac{z(t-\iota)}{z(t_*)}=\infty. \tag{2.68}$$

Dividing (2.66) by $z(t_*)$ we have

$$\frac{z(t)}{z(t_*)}+\frac{z(t-\iota)}{z(t_*)}\frac{\beta\alpha}{c+2\gamma}\left[e^{-(c+2\bar{\gamma})\tau}-e^{-(c+2\bar{\gamma})\iota}\right]\leq 1,\ \ t\geq t_1 \tag{2.69}$$

which contradicts (2.68) and (2.53). $\square$

The proof of Lemma 5(iii) is similar to that of 5(i).

Following Lemma 2.3, the delayed differential equation (2.43) has a non-oscillatory solution if $\beta<0$ and $\frac{\beta\tau}{c+2\bar{\gamma}}\left[e^{-(c+2\bar{\gamma})\tau}-1\right]\leq\frac{1}{e}$. Under these condition, we can describe the asymptotic behaviors of solutions of (2.43). Moreover, we seek a solution in the form $u(t)=\psi_1+\psi_2e^{\rho t}$, where ψ_1, ψ_2 and ρ are arbitrary constants. In this case, the characteristic equation with respect to (2.43) is

$$h(\rho) = \rho - c - \beta \left[\frac{1 - e^{-(\rho+2\bar{\gamma})\tau}}{\rho + 2\bar{\gamma}} \right] = 0. \tag{2.70}$$

From $u(t_0) = u_0$, we obtain

$$\begin{aligned} \psi_1 &= u^* = - \left[\frac{\nu}{c + \frac{\beta}{2\bar{\gamma}}(1 - e^{-2\bar{\gamma}\tau})} \right], \\ \psi_2 &= (u_0 - \psi_1) e^{-\rho t_0}. \end{aligned} \tag{2.71}$$

However, using numerical simulation for (2.43), we observe that $u(t)$ is asymptotically stable. From (2.46) and (4.43), the numerical scheme is defined as follows;

$$\begin{aligned} v_i &= (1 + c\Delta t + \beta(\Delta t)^2 e^{-2\bar{\gamma}}) v_{i-1} + \beta(\Delta t)^2 (v_{i-2} e^{-4\bar{\gamma}} + v_{i-3} e^{-6\bar{\gamma}} + ... + v_{i-l} e^{-2\bar{\gamma}} l) \\ u_i &= v_i + u^* \end{aligned} \tag{2.72}$$

where $v_i = v(t_i)$, $u_i = u(t_i)$ and $\{t_i\}_{i=1}^{m}$ is the time grid with a mesh of constant size Δt, l is the discrete-time delay analogue of τ.

Solution is shown in Figure (2).

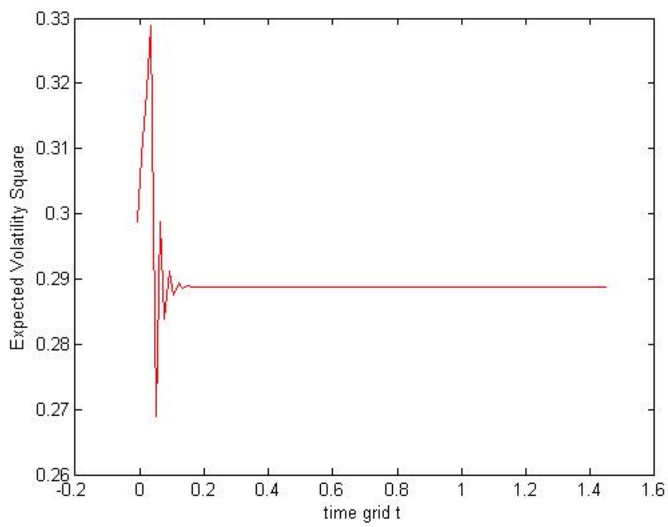


Figure 2.: Solution of (2.43) with parameters in Table 2.

Chapter 3
Parameter Estimation

3.1 Introduction

In this chapter, we find an expression for the forward price of energy commodity. Using the representation of forward price, we apply the Least-Square Optimization and Maximum Likelihood techniques to estimate the parameters defined in (2.2) and (2.34).

3.2 Derivation of Forward Price

Let $F(t,T)$ be the forward price at time t of an energy goods with maturity at time T. We define

$$F(t,T) = \mathbb{E}_{\bar{\mathcal{P}}}\left(S(T)\right) \tag{3.1}$$

where $S(T)$ is defined by (2.1), the expectation here is taken with respect to the risk neutral measure defined in (2.30).

REMARK 5 At maturity, it is expected that the forward price is equal to the spot price at that time i.e $F(T,T) = S(T)$. This is the basic assumption of the risk neutral valuation method.

From (2.34), the forward price $F(t,T)$ can be expressed as

$$\begin{aligned} F(t,T) &= \mathbb{E}_{\bar{\mathcal{P}}}\left(S_T\right) \\ &= \mathbb{E}_{\bar{\mathcal{P}}}\left(\exp[f(T) + x_2(T)]\right) \\ &= \exp\left[f(T) + e^{-\bar{\gamma}(T-t)}x_2(t) + \bar{\phi}(t,T)x_1(t) + \bar{\omega}(t,T) + \Upsilon(t,T)\right], \end{aligned} \tag{3.2}$$

where $\Upsilon(t,T)$ is define by

$$\Upsilon(t,T) = \exp\left[\frac{\psi_1 g(t,T,2\bar{\gamma}) + (u_0 - \psi_1)e^{\rho T} g(t,T,\rho + 2\bar{\gamma}) + \left[\frac{\delta\bar{\gamma}}{\bar{\mu}-\bar{\gamma}}\right]^2 h(t,T,\bar{\mu},\bar{\gamma})}{2}\right],$$

and

$$h(t,T,\bar{\mu},\bar{\gamma}) = \left(g(t,T,2\bar{\gamma}) - 2g(t,T,\bar{\mu}+\bar{\gamma}) + g(t,T,2\bar{\mu})\right).$$

$$g(t,T,a) = \frac{1-e^{-a(T-t)}}{a}, \text{for any } a \in \mathbb{R} \tag{3.3}$$

and ψ_1 is defined in (2.71). Hence

$$\begin{aligned} \log F(t,T) &= f(T) + e^{-\bar{\gamma}(T-t)}x_2(t) + \bar{\phi}(t,T)x_1(t) + \bar{\omega}(t,T) + \Upsilon(t,T) \\ &= f(T) + e^{-\bar{\gamma}(T-t)}\left(\log S(t) - f(t)\right) + \bar{\phi}(t,T)x_1(t) + \bar{\omega}(t,T) + \Upsilon(t,T) \\ &= A(t,T) + B(t,T)x_1(t) \end{aligned} \tag{3.4}$$

where $A(t,T) = f(T) + e^{-\bar{\gamma}(T-t)}\left(\log S(t) - f(t)\right) + \bar{\omega}(t,T) + \Upsilon(t,T)$ and $B(t,T) = \bar{\phi}(t,T)$. Define

$$\begin{aligned} \epsilon_1 &= (\bar{\mu}, \bar{\kappa}, \delta) \\ \epsilon_2 &= (\bar{\gamma}, \alpha, \beta, c, \tau) \\ \epsilon_3 &= (A_0, A_1, A_2, B_1, B_2) \\ \epsilon &= (\epsilon_1, \epsilon_2, \epsilon_3), \end{aligned} \tag{3.5}$$

where ϵ consists of the risk-neutral parameters in (2.2) and (2.34).

We can represent $\log F(t,T)$ as $\log F(t,T;\epsilon)$, $x_1(t) \equiv x_1(t;\epsilon_1)$, $x_2(t) \equiv x_2(t;\epsilon_2)$, $f(t) \equiv f(t;\epsilon_3)$.

In the following section, we use the Least square optimization approach to estimate the parameters $\bar{\gamma}, \bar{\mu}, \bar{\kappa}$ and δ.

3.3 Parameter Estimation Techniques

In this section, we discuss about the estimation of parameters of the stochastic interconnected models for energy commodity's spot price (2.14). A numerical scheme is used to develop time-series model, and using the Least Squares optimization and Maximum Likelihood techniques, we outline the parameter estimations for our model.

3.3.1 Least Squares Optimization Techniques

For time t_i, $i \in \{1, 2, ..., N\} = I(1, N)$, let $S(t_i)$ denote the historical spot price of commodity. For fixed $i \in I(1, N)$, $\tilde{F}(t_i, T_j^i)$ represent an observe future price at a time t_i with delivery time T_j^i for $j \in I(1, n_i)$. These data values are obtainable from the energy market.

For each given quoted time t_i, we obtain $x_1(t;\epsilon_1)$ such that it minimizes the sum of squares

$$\text{sqdiff}(t_i, \epsilon) = \sum_{j=1}^{n_i} \left[\log\ F(t_i, T_j^i; \epsilon) - \log \tilde{F}(t_i, T_j^i)\right]^2, \tag{3.6}$$

where $\log\ F(t_i, T^i_j; \epsilon)$ is described in (3.4). Differentiating (3.6) with respect to $x_1(t; \epsilon_1)$ and setting the result to be zero to get the optimal value of $x_1(t; \epsilon_1)$ as a function of the parameter set, we have

$$\tilde{x}_1(t_i; \bar{\epsilon}) = \frac{\sum_{j=1}^{n_i} \left[B(t_i, T^i_j) \left(\log \tilde{F}(t_i, T^i_j) - A(t_i, T^i_j) \right) \right]}{\sum_{j=1}^{n_i} \left[B(t_i, T^i_j) \right]^2}, \ \ i \in I(1, m), \tag{3.7}$$

Substituting this optimal value into the initial sum of squares (3.6), and summing over the range of initial times $\{t_i\}$ and performing a nonlinear least-squares optimization as follows:

$$\text{sqdiff}(\epsilon) = arg \min_{\epsilon} \sum_{i=1}^{N} \sum_{j=1}^{n_i} \left[A_{t_i, T^i_j} + B_{t_i, T^i_j} x_1(t)(\bar{\epsilon}) - \log \tilde{F}(t_i, T^i_j) \right]^2. \tag{3.8}$$

With the obtained $\bar{\epsilon}$, $\{x_1(t)\}^N_1$, $\{x_2(t)\}^N_1$, $\{f(t)\}^N_1$ and $\{S_t\}^N_1$ are easily computed.

In the case of real-world $\mathcal{P}$-parameters $[\gamma, \mu, \kappa, \delta]$ estimation, the estimates of γ and κ are obtained using a linear regression technique associated with the model $dx_2 = \gamma(\kappa - x_2(t))dt + \sigma dW_2(t)$. (3.7) contains an estimated hidden process $\bar{x_1}(t_i)$ which is obtained by the least square minimization approach. This estimated data is used in a regression of a one-factor mean reverting model $dx_1(t) = \mu(\kappa - x_1(t)) + \delta dW_1(t)$ to obtain estimates for μ and δ. We remark that this procedure is very stable.

3.3.2 Maximum Likelihood Approach

Now, by following the approach in [140] and using Maximum Likelihood approach, the time delay and the delay volatility parameters α, β and c are estimated. Our model contains two sources of randomness, that is, the Wiener process in the equation for log of spot price and another Wiener process in the equation for expected log of spot price. Therefore, the presented model is an extension of GARCH model [140]

An outline of the procedure is given below. From (2.40), we have that

$$\frac{d\sigma^2(t, x_2)}{dt} = \alpha + \beta \left[\int_{t-\tau}^{t} \sigma(s, x_2) e^{-\bar{\gamma}(t-s)} d\bar{W}_2(s) + \delta \int_{t-\tau}^{t} \bar{\phi}(u, t) d\bar{W}_1(u) \right]^2 + c\sigma^2(t, x_2). \tag{3.9}$$

We define the discrete-time analogue value l to the continuous-time delay τ as $l = \left[|\frac{\tau}{\Delta}| \right]$, where $[|.|]$ is the floor function, Δ is the size of the mesh of the discrete-time grid. Hence, we define

$$\begin{aligned} \varepsilon_i &= \sigma_i \xi_i \\ \eta_i &= \delta \zeta_i, \end{aligned} \tag{3.10}$$

where $\xi,\ \zeta$ are standard normal variate. The discrete-time delayed model corresponding to (3.9) for volatility is described by

$$\sigma_n^2 = \alpha + \beta\Delta t\left[\sum_{i=1}^{l}\left(\varepsilon_{n-i}e^{-\bar{\gamma}i} + \eta_{n-i}\bar{\phi}(0,i)\right)\right]^2 + r\sigma_{n-1}^2, \tag{3.11}$$

where $n = 1, 2, 3, 4, ...$, and $r = 1 + c$.

From (2.39), we further note that

$$\begin{aligned}\int_{t-\tau}^{t}\sigma(s,x_2(s))e^{-\bar{\gamma}(t-s)}d\bar{W}_2(s) + \delta\bar{\phi}(s,t)d\bar{W}_1(s) &= x_2(t) - x_2(t-\tau)e^{-\bar{\gamma}\tau} \\ &\quad -x_1(t-\tau)\bar{\phi}(0,\tau) - \bar{\omega}(0,\tau),\end{aligned}$$

that is,

$$\sqrt{\Delta t}\sum_{i=1}^{l}\varepsilon_{n-i}e^{-\bar{\gamma}i} + \eta_{n-i}\bar{\phi}(0,i) = x_2(n) - x_2(n-1)e^{-\bar{\gamma}l} - x_1(n-1)\bar{\phi}(0,l) - \bar{\omega}(0,l). \tag{3.12}$$

Define

$$P(n) = \left[x_2(n) - x_2(n-1)e^{-\bar{\gamma}l} - x_1(n-1)\bar{\phi}(0,l) - \bar{\omega}(0,l)\right]^2, \tag{3.13}$$

This together with (6.1) yields

$$\sigma_n^2 = \alpha + \beta P_n + r\sigma_{n-1}^2. \tag{3.14}$$

The solution of difference equation (3.14) is given by

$$\sigma_n^2 = \begin{cases}\alpha F_n(r) + \beta G_n(r) + H_n(r), & n \geq l+1 \\ \varepsilon_n^2 & n \leq l,\end{cases} \tag{3.15}$$

where for $r = 1 + c$,

$$F_n = \sum_{i=0}^{n-l-1} r^i, \tag{3.16}$$

$$G_n = \sum_{i=0}^{n-l-1} r^i P_{n-i}, \tag{3.17}$$

$$H_n = r^{n-l}\sigma_l^2. \tag{3.18}$$

We observe that the series F_n in (3.16) converges if $|r| < 1$, that is, $|1 + c| < 1$. Hence,

$$-2 < c < 0. \tag{3.19}$$

From the definition of ε_n in (3.10), the probability density function f_{ε_n} of ε_n is

$$f_{\varepsilon_n}(y) = \frac{1}{\sqrt{2\pi}\sigma_n}\exp\left[-\frac{y^2}{2\sigma_n^2}\right]. \tag{3.20}$$

Thus the likelihood function $L(\alpha, \beta, c)$ of $f_\varepsilon,\ n \in I(1, N)$ for arbitrary large positive integer N is

$$L(\alpha, \beta, c) = \prod_{n=1}^{N} \frac{1}{\sqrt{2\pi}\sigma_n} \exp\left[-\frac{y^2}{2\sigma_n^2}\right]. \tag{3.21}$$

By applying the Maximum Likelihood method, we obtain the estimates $\alpha(l)$, $\beta(l)$, and $r(l)$ for $l \in I(1, p)$ for some arbitrary p.

3.4 Some Results: Natural Gas

In this section, we apply our model to the Henry Hub daily natural gas data set for the period $02/01/2001$-$09/30/2004$ [25]. The data is collected from the United State Energy Information Administration website (www.eia.gov). Using the Henry Hub daily natural gas data set, we present the calibration results of our model. The parameter estimates of our model for the value of $l = 2$ are given. For this purpose, using a combination of direct search method and Nelder-Mead simplex algorithm, we search iteratively to find the parameters that maximizes the likelihood function. All codes are written in Matlab.

Table 2: Estimated Parameters of Henry Hub daily natural gas spot prices [25] for the period 02/01/2001-09/30/2004.

$\bar{\gamma}$	$\bar{\mu}$	$\bar{\kappa}$	δ	τ	α	β	c
1.8943	1.0154	1.5627	0.36	0.008	0.433	-0.07	-1.5

Table 2 shows the risk-neutral parameter estimates of Henry Hub daily natural gas data set [25] for the period 02/01/2001-09/30/2004.

The next figure shows the graphs of the real spot natural gas price data set [25] together with the simulated spot price $S(t)$.

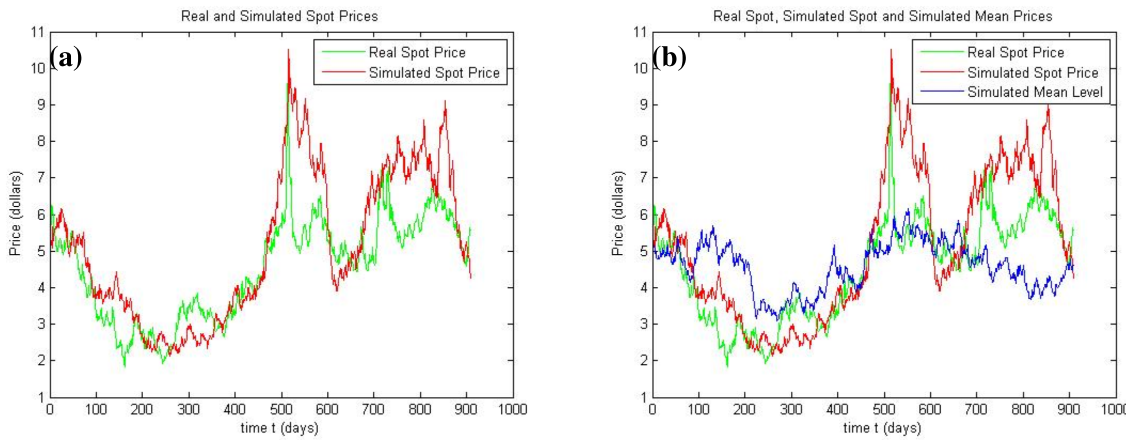


Figure 3.: Real, Simulated and Forecasted Prices.

Figure 3 (a) shows the graphs of the real spot natural gas price data set [25] together with the simulated spot price $S(t)$. Figure 3 (b) shows and the the graphs of the real spot natural price data set together with the simulated spot price $S(t)$ and the simulated expected spot price $\exp(x_1(t))$. We notice that in Figure 3 (a), the simulated spot price captures the dynamics of the data set. This shows that the simulation agrees with our mathematical model. Another observation is that the simulated mean level seems to move around the value 4.80 which is close to $\exp(\bar{\kappa}) = \exp(1.5627)$. This confirms the fact that $\bar{\kappa}$ is the equilibrium mean level.

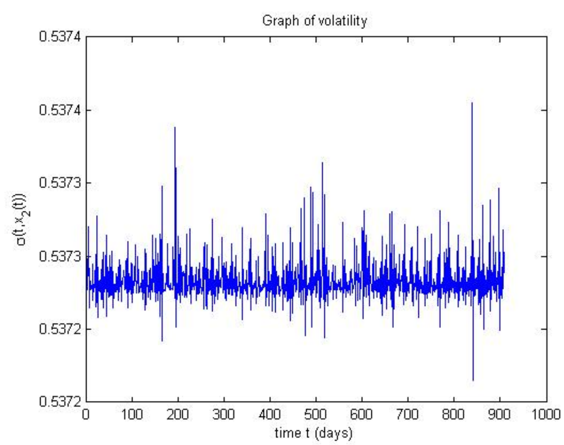


Figure 4.: Simulated $\sigma(t, x_2(t))$.

Figure 4 shows the plot of volatility $\sigma(t, x_2(t))$ with time. It is clear from the graph that the solution is non-oscillatory. This is because Lemma 2.3 (i) is satisfied using the parameter estimates in Table 2.

Chapter 4

Non-Linear Stochastic Modeling of Energy Commodity Spot Price Processes with Delay in Volatility

4.1 Introduction

In real world situation, the expected spot price of energy commodities and its measure of variation are not constant. This is because of the fact that a spot price is subject to both deterministic and random environmental perturbations. Moreover, some statistical studies of stock prices [8] raised the issue of market's delayed response. This indeed causes the price to drift significantly away from the market quoted price. It is well recognized that time-delay models in economics [41] are more realistic than the models without time-delay. Continuous-time and Discreet-time stochastic volatility models [9, 38] have been developed in economics. Elloit et al [37] developed a model for pricing variance swaps and volatility swaps under a continuous-time Markov-modulated version of Heston's stochastic volatility model. Recently, in a survey work, Hansen and Lunde [46] have estimated these types of models and concluded that the performance of the GARCH(1,1) model is better than any other model. Furthermore, Cox-Ingersoll-Ross(CIR) developed a mean reverting interest rate model that was based on the mean-level interest rate with exponentially weighted integral of its past history, the relationship between level dependent volatility and the square root of the interest rate [19]. Employing the Ornstein Uhlenbeck [126] and Cox-Ingersoll-Ross(CIR) [19] processes, Heston developed a stochastical model for the volatility of stock spot asset.

In this work, using basic economic principles, we systematically develop interconnected stochastic nonlinear dynamic model for the log-spot price, expected log-spot price and volatility processes. The effort is made to utilize the developed interconnected stochastic model to analyze the Henry Hub daily natural gas data set. The by-product of this led to the development of discretized expected square volatility model and a modification of The Kalman filter approach. This has been achieved by treating a diffusion coefficient parameter in the non-seasonal log-spot price dynamic system as a stochastic volatility functional of log-spot price.

The organization of this study is as follows:

In Section 4.2, we developed a stochastic models for energy commodity's spot price. We extend the linear interconnected deterministic and stochastic models in (2.14) to non-linear interconnected deterministic and stochastic models. In Section 4.3, the derived model is validated. In Section 4.5, by outlining the risk-neutral dynamics and pricing, risk-neutral dynamics of presented model is derived.

4.2 Model Derivation

The principles of demand and supply processes suggest that the price of a energy commodity will remain within a given finite lower and upper bounds. Let $\kappa_1 \geq 0$ and $\kappa_2 > 0$ be the expected lower and upper limits of the nonseasonal log of spot price, respectively. In a real world situation, the nonseasonal log of spot price is governed by the spot price dynamic process. In the following, we outline the development of dynamic model for the nonseasonal spot price processes. Let $x_2(t)$ be the nonseasonal log of spot price at a time t. In this case, κ_2 characterizes the fixed cost, $(x_2(t) + \kappa_1)(\kappa_2 - x_2(t))$ characterizes the market potential for $x_2(t)$ per unit of time at a time t. The market potential is induced/generated by the underlying market forces on the nonseasonal $\log$ spot price, $x_2(t)$. This leads to the following principle regarding the dynamic of price $x_2(t)$ of energy goods. The change in nonseasonal log spot price of the energy commodity $\Delta x_2(t) = x_2(t + \Delta t) - x_2(t)$ over the interval of length $|\Delta t|$ is directly proportional to the product of the market potential price and the length of the interval.

$$\Delta x_2(t) \propto (x_2(t) + \kappa_1)(\kappa_2 - x_2(t))\Delta t. \tag{4.1}$$

This implies

$$dx_2(t) = \gamma(x_2(t) + \kappa_1)(\kappa_2 - x_2(t))dt, \tag{4.2}$$

where γ is a positive constant of proportionality, dx_2 and dt are differentials of $x_2(t)$ and t, respectively.

We note that (4.2) has a unique non-zero equilibrium κ_2. Moreover, we observe that whenever the price lies above κ_2, there is a tendency for the price to fall and whenever the price is below κ_2, the price rises back. Hence, κ_2 is the equilibrium of (4.2). Hence

$$\lim_{t\to\infty} x_2(t) = \kappa_2 \tag{4.3}$$

In the real world situation, the upper price limit of the nonseasonal log spot price κ_2 is not a constant parameter. In the following, we employ the argument of Bernard and Thomas [8] to incorporate both the response time delay and random environmental perturbations into the measure of variation of the log-spot price process of energy commodity. Therefore, we consider

$$\kappa_2 = x_1(t) + e_2(t), \tag{4.4}$$

where e_2 is a white noise process that characterizes the measure of random variation of the log spot price, $x_1(t)$ describes a mean of non-seasonal log spot price process and it is assumed to be governed by a similar differential equation described in (4.2), that is,

$$dx_1(t) = \mu(x_1(t) + \kappa_3)(\kappa_2 - x_1(t))dt, \tag{4.5}$$

where μ is a positive constant of proportionality.

Moreover, the mean non-seasonal log spot process is subject to random environmental perturbations. By following the argument used in (4.4), we assume that κ_3 is subject to random perturbations:

$$\kappa_3 = \kappa_0 + e_1, \tag{4.6}$$

where κ_0 is constants, and e_1 is a white noise and it describes the measure of random influence on the mean non-seasonal log-spot price.

Substituting (4.4) and (4.6) into (4.2) and (4.5), respectively, we obtain

$$\begin{cases} dx_1(t) &= \mu(x_1(t) + \kappa_0)(\kappa_2 - x_1(t))dt + \mu(\kappa_2 - x_1(t))e_1(t)dt, \\ dx_2(t) &= \gamma(x_2(t) + \kappa_1)(x_1(t) - x_2(t))dt + \gamma(x_2(t) + \kappa_1)e_2(t)dt. \end{cases} \tag{4.7}$$

Using (4.7) and following the argument of Bernard and Thomas [8], we incorporate both the response time delay and random environmental perturbations into the measure of random variations on the log-spot price process of energy commodity. This leads to the establishment of a stochastic model for nonseasonal log spot price and expected log-spot price processes that is described by the following non-linear system of stochastic functional differential equations:

$$\begin{cases} dx_1 &= \mu(x_1 + \kappa_0)(\kappa_2 - x_1)dt + \delta(\kappa_2 - x_1)dW_1(t), \;\; x_1(t_0) = x_{10}, \\ dx_2 &= \gamma(x_2 + \kappa_1)(x_1 - x_2)dt + \sigma(t, x_{2t})(x_2 + \kappa_1)dW_2(t), \;\; x_{t_{02}} = \vartheta_{02}, \end{cases} \tag{4.8}$$

where

$$\begin{cases} \mu e_1(t)dt &\equiv \delta dW_1(t) \\ \gamma e_2(t)dt &\equiv \sigma(t, x_{2t})dW_2(t), \end{cases} \tag{4.9}$$

and $\delta > 0$, x_{2t} is a segment of continuous function x_2 defined by $x_{2t}(\theta) = x_2(t+\theta)$, $\theta \in [-\tau, 0]$ for $t \geq 0$, σ is a functional defined on $[0,T] \times \mathcal{C}[[-\tau,0],\mathbb{R}]$ into $\mathbb{R}$.

For the sake of validity and completeness of mathematical model (4.8), we assume the following:

$\mathbf{H}_1$: $x_{2t}(\theta) = x_2(t+\theta)$, $\theta \in [-\tau, 0]$, $x_t = \vartheta \in C[[-\tau,0],\mathbb{R}^2]$ defined as $x_t(\theta) = x(t+\theta) = [x_1(t+\theta), x_2(t+\theta)]^T$, $\tau \geq 0, \gamma > 0, \mu > 0, \kappa_1 \geq 0, \kappa_2 > 0, \kappa_0 \geq 0, \delta > 0$, $\sigma : [0,T] \times \mathcal{C} \to \mathbb{R}_+$ is a Lipschitz continuous bounded mapping, $\mathcal{C}$ is the Banach space of continuous functions defined on $[-\tau, 0]$ into $\mathbb{R}$ equipped with the supremum norm; $W_1(t)$ and $W_2(t)$ are standard Wiener process on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{t\geq 0}, \mathcal{P})$, the filtration function $(\mathcal{F})_{t\geq 0}$ is right-continuous, and each $\mathcal{F}_t$ with $t \geq 0$ contains all $\mathcal{P}$-null sets in $\mathcal{F}$.

By following the idea of [140], we define the continuous volatility version of the GARCH type model as:

$$d\sigma^2(t,\vartheta_2) = \left(\alpha + c\sigma^2(t,\vartheta_2) + \beta\left[\int_{t-\tau}^{t} \sigma(s,\vartheta_2)dW_3(s)\right]^2\right)dt, \;\; \sigma^2(t_0,\vartheta_{02}) = \sigma_0^2(\vartheta_{02}) \quad (4.10)$$

where $\alpha, \beta \in \mathbb{R}_+$, $c < 0$, and W_3 is a Wiener process..

From (4.8) and (4.10), the overall stochastic dynamic model for nonseasonal log spot price, expected log-spot price and volatility processes under random perturbation is described by the following non-linear system of stochastic functional differential equations

$$\begin{cases} dx_1 &= \mu(x_1+\kappa_0)(\kappa_2 - x_1)dt + \delta(\kappa_2 - x_1)dW_1(t), \;\; x_1(t_0) = x_{10}, \\ dx_2 &= \gamma(x_2+\kappa_1)(x_1 - x_2)dt + \sigma(t,x_{2t})(x_2+\kappa_1)dW_2(t), \;\; x_{t_{02}} = \vartheta_{02}, \\ d\sigma^2(t,\vartheta_2) &= \left(\alpha + c\sigma^2(t,\vartheta_2) + \beta\left[\int_{t-\tau}^{t}\sigma(s,\vartheta_2)dW_3(s)\right]^2\right)dt, \\ & \quad \sigma^2(t_0,\vartheta_{02}) = \sigma_0^2(\vartheta_{02}). \end{cases} \quad (4.11)$$

4.3 Mathematical Model Validation

In this section, we validate the mathematical model derived in Section 2. We note that the classical existence and uniqueness theorem is not directly applicable to (4.8). We need to modify the existence and uniqueness results. The modification is based on Theorem 3.4 [57] and the usage of linear invertible transformation. For this, we first transform the systems of nonlinear stochastic system of differential equations (4.8) into a geometric mean reverting non-linear stochastic systems of differential equations. We show the global existence of solution process of transformed systems of differential equations. From this, the solution of the geometric mean reverting non-linear stochastic system follows immediately.

LEMMA 4.1 *Using the transformation*

$$a). \quad \begin{cases} y_1 &= x_1 - \kappa_2 \\ y_2 &= x_2 + \kappa_1, \end{cases} \tag{4.12}$$

$$b). \quad \begin{cases} \varphi_1 &= \vartheta_1 - \kappa_2 \\ \varphi_2 &= \vartheta_2 + \kappa_1, \end{cases} \tag{4.13}$$

we have $dy_i(t) = dx_i(t)$ and hence, the system of (4.11) is reduced to

$$\begin{cases} dy_1 &= -\mu y_1[\lambda_1 + y_1]dt - \delta y_1 dW_1, \quad y_1(t_0) = y_{10}, \\ dy_2 &= \gamma y_2[\lambda_2 + y_1 - y_2]dt + \sigma(t, y_{2t} - \kappa_1)y_2 dW_2, \quad y_{t_{02}} = \varphi_{02} \\ d\sigma^2(t, \varphi_2 - \kappa_1) &= \left(\alpha + c\sigma^2(t, \varphi_2 - \kappa_1) + \beta\left[\int_{t-\tau}^{t} \sigma(s, \varphi_2 - \kappa_1)dW_3(s)\right]^2\right)dt, \\ & \sigma^2(t_0, \varphi_{02} - \kappa_1) = \sigma_0^2(\vartheta_{02}), \end{cases} \tag{4.14}$$

where

$$\begin{cases} \lambda_1 &= \kappa_0 + \kappa_2 \\ \lambda_2 &= \kappa_1 + \kappa_2. \end{cases} \tag{4.15}$$

In the following, we give the existence and uniqueness conditions for solutions of the IVP (4.14).

We recall that system of stochastic differential equations (4.14) does not satisfy the classical existence and uniqueness conditions. However it does satisfy the local Lipschitz condition. We construct sequences of functions for the drift and volatility parts of (4.14) such that the classical existence theorem conditions are valid for a sequence of modified rate coefficients defined on a cylinder $[t_0, \infty) \times U_n$, for $t_0 \in \mathbb{R}$, $n \in \{1, 2, 3, ...\}$, where U_n are modified sequence of rate functions defined as:

$$U_n = \{|y|_0 < n\}, \tag{4.16}$$

$$\begin{cases} b_1(t, \varphi_1(0)) &= -\mu\varphi_1(0)(\kappa_0 + \kappa_2 + \varphi_1(0)) \\ b_2(t, \varphi_2(0)) &= \gamma\varphi_2(0)(\varphi_1(0) - \varphi_2(0) + \kappa_1 + \kappa_2) \\ b_3(t, \varphi_2) &= \alpha + c\sigma^2(t, \varphi_2 - \kappa_1) + \beta\left[\int_{t-\tau}^{t} \sigma(s, \varphi_2 - \kappa_1)dW_3(s)\right]^2 \\ \sigma_1(t, \varphi_1(0)) &= -\delta\varphi_1(0) \\ \sigma_2(t, \varphi_2) &= \sigma(t, \varphi)\varphi_2(0) \end{cases} \tag{4.17}$$

where $\varphi = (\varphi_1, \varphi_2)^T$; $|\varphi|_0 = \sup_{-\tau \le s \le 0} |\varphi(s)|$, $\varphi_i \in C[[-\tau, 0], \mathbb{R}^2]$, for $i = 1, 2$, and

$$b_1^{(n)}(t, \varphi_1(0)) = \begin{cases} b_1(t, \varphi_1(0)) \ \text{ for } \ |\varphi|_0 < n \\ b_1(t, n) \ \text{ for } \ |\varphi| \ge n \end{cases} \tag{4.18}$$

$$\sigma_1^{(n)}(t, \varphi_1(0)) = \begin{cases} \sigma_1(t, \varphi_1(0)) \ \text{ for } \ |\varphi|_0 < n \\ \sigma_1(t, n) \ \text{ for } \ |\varphi|_0 \ge n, \end{cases} \tag{4.19}$$

$$b_2^{(n)}(t, \varphi_2(0)) = \begin{cases} b_2(t, \varphi_2(0)) \ \text{ for } \ |\varphi|_0 < n \\ b_2(t, n) \ \text{ for } \ |\varphi|_0 \ge n \end{cases} \tag{4.20}$$

$$\sigma_2^{(n)}(t, \varphi_2) = \begin{cases} \sigma_2(t, \varphi) \ \text{ for } \ |\varphi|_0 < n \\ \sigma_2(t, n) \ \text{ for } \ |\varphi|_0 \ge n, \end{cases} \tag{4.21}$$

$$b_3^{(n)}(t, \varphi_2) = b_3(t, \varphi_2) \ \forall \ n \tag{4.22}$$

Using the sequence of functions $(4.18 - 4.22)$, the modified system of stochastic differential equations (4.14) is described by

$$\begin{cases} dy_1^{(n)} & = & b_1^{(n)}(t, y_1^{(n)})dt + \sigma_1^{(n)}(t, y_1^{(n)})dW_1, \ \ y_1(t_0) = \varphi_1(0), \\ dy_2^{(n)} & = & b_2^{(n)}(t, y_2^{(n)})dt + \sigma_2^{(n)}(t, y_{2t}^{(n)})dW_2, \ \ y_2(t_0) = \varphi_{02}, \\ d\sigma^2(t, \vartheta)^{(n)} & = & b_3^{(n)}(t, y_2^{(n)})dt, \ \ \sigma^2(t_0, \varphi_{02}) = \sigma_0^2(\varphi_{02}). \end{cases} \tag{4.23}$$

Hence, from (4.18)-(4.22) and assumption $\mathbf{H}_1$, system (4.23) satisfies the classical existence and uniqueness conditions [57]. Therefore, there exist a sequence of Markov process $y_1^{(n)}$ and $y_2^{(n)}$ corresponding to equation (4.23). Next, we show that the global solution of (4.14) exists. For this purpose, we need to utilize the following concepts.

DEFINITION 4.3.1 *Define $\tau_1^{(n)}$ and $\tau_2^{(n)}$ to be the first exit time of the process $y_1^{(n)}(t)$ and $y_2^{(n)}(t)$ from the set $|y_1| < n$ and $|y_2| < n$ respectively, that is*

$$\tau_i^{(n)} = \inf\{t > 0 \ : \ |y_i(t)| \ge n\}, \ \ i = 1, 2. \tag{4.24}$$

Define τ_1 and τ_2 to be the (finite or infinite) limit of the monotone increasing sequence $\tau_1^{(n)}$ and $\tau_2^{(n)}$ respectively as $n \to \infty$.

$$\tau_i = \lim_{n \to \infty} \tau_i^{(n)} = \inf\{t > 0 \ : \ |y_i(t)| \notin [0, \infty)\}, \ \ i = 1, 2. \tag{4.25}$$

A process $X(t)$ is regular if for any $(s, x) \in I \times \mathbb{R}^l$,

$$\boldsymbol{P}\{\tau = \infty\} = 1 \tag{4.26}$$

where τ is the limit of the first exit time τ_n.

Using Theorems 3.4 and 3.5 of [57], we show that the process $\mathbf{y}(t) = \{y_1(t), y_2(t)\}$ is regular. To do this, we cite the Theorem and show that the conditions in the Theorem are satisfied.

THEOREM 4.1 *(Theorem 3.5) [57] Suppose that the local solution of (4.23) exists on every cylinder $[t_0, \infty) \times U_n$ and, moreover, that there exists a nonnegative function $V \in \mathbb{C}_2$ such that for some constant $c > 0$*

$$\begin{cases} \boldsymbol{L}V \leq cV \\ V_n = \inf_{|y|>n} V(t,y) \to \infty \;\; as \;\; n \to \infty, \end{cases} \tag{4.27}$$

*where the **L**-operator is given by*

$$\boldsymbol{L} = \frac{\partial}{\partial t} + \sum_{i=1}^{l} b_i^{(n)}(t, y^{(n)}) \frac{\partial}{\partial y^{(n)}} + \frac{1}{2} \sum_{i,j=1}^{l} \sigma_i^{(n)} \sigma_j^{(n)}(t, y^{(n)}) \frac{\partial^2}{\partial y_i^{(n)} \partial y_j^{(n)}}. \tag{4.28}$$

Then, for every random variable $x(t_0)$ independent of the process $W_i(t) - W_i(t_0)$ there exists a solution $y(t)$ of the system of stochastic differential equation (4.14) which is almost surely continuous stochastic process and is unique up to equivalence.

Proof. We utilize the structure of system (4.23) and establish the conclusion of the theorem for the first component of (4.23), followed by the second component by knowing the nature of the third component of (4.23) in Appendix A.1.

We define a new stochastic process $\tilde{y}_1(t)$ as

$$\tilde{y}_1 = y_1^{(n)}, \;\; \text{for} \;\; t \leq \tau_1^{(n)}. \tag{4.29}$$

We show that condition (4.26) is satisfied for y_1, thereby making the process $y_1(t)$ to be almost surely defined for all $t > t_0$.

We define a nonnegative function V_1 on $E = [t_0, \infty) \times \mathbb{R}_+$ into $\mathbb{R}^+$ as follows;

$$V_1(t, y_1) = \int_0^{y_1} (u^2+1)^{\frac{\mu(\kappa_0+\kappa_2)}{\delta^2}} du + \frac{\delta^2}{\kappa_0+\kappa_2} \left[\frac{\mu(\kappa_0+\kappa_2)(1+\kappa_0+\kappa_2)}{\delta^2+\mu(\kappa_0+\kappa_2)} \right]^{\frac{\mu(\kappa_0+\kappa_2)}{\delta^2}+1}. \tag{4.30}$$

It is obvious that $V_1 \in \mathcal{C}_{1,2}$. Moreover, the **L**-operator with respect to the first component of system of stochastic differential equation (4.14) satisfy

$$\begin{aligned} \mathbf{L}V_1 &= -\mu(\kappa_0+\kappa_2) y_1 (y_1^2+1)^{\frac{\mu(\kappa_0+\kappa_2)}{\delta^2}-1} - \mu y_1^2 (y_1^2+1)^{\frac{\mu(\kappa_0+\kappa_2)}{\delta^2}} \\ &\leq \mu(\kappa_0+\kappa_2)(y_1^2+1)(y_1^2+1)^{\frac{\mu(\kappa_0+\kappa_2)}{\delta^2}-1} - \mu y_1^2 (y_1^2+1)^{\frac{\mu(\kappa_0+\kappa_2)}{\delta^2}} \\ &= \mu(\kappa_0+\kappa_2-y_1^2)(y_1^2+1)^{\frac{\mu(\kappa_0+\kappa_2)}{\delta^2}}. \end{aligned}$$

Case 1: If $\kappa_0 + \kappa_2 - y_1^2 \leq 0$, then $LV_1 \leq 0 \leq V_1$.

Case 2: If $\kappa_0 + \kappa_2 - y_1^2 > 0$, then $-\sqrt{\kappa_0 + \kappa_2} < y_1 < \sqrt{\kappa_0 + \kappa_2}$ and

$$\mu(\kappa_0 + \kappa_2 - y_1^2)(y_1^2 + 1)^{\frac{\mu(\kappa_0+\kappa_2)}{\delta^2}} < \frac{\delta^2}{\kappa_0 + \kappa_2}\left[\frac{\mu(\kappa_0 + \kappa_2)(1 + \kappa_0 + \kappa_2)}{\delta^2 + \mu(\kappa_0 + \kappa_2)}\right]^{\frac{\mu(\kappa_0+\kappa_2)}{\delta^2}+1}, \tag{4.31}$$

since the function $f(x) = \mu(\kappa_0 + \kappa_2 - x^2)(x^2 + 1)^{\frac{\mu(\kappa_0+\kappa_2)}{\delta^2}}$ has a maximum point at $x = \sqrt{\kappa_0 + \kappa_2 - \frac{(\kappa_0+\kappa_2)\delta^2}{\mu(\kappa_0+\kappa_2)+\delta^2}}$.

Hence, $\mathbf{L}V_1 \leq V_1$.

Thus, in both cases,

$$\mathbf{L}V_1 \leq V_1. \tag{4.32}$$

Furthermore,

$$\begin{cases} V_{1_n} & = \inf\limits_{|y_1|>n} V_1(t, y_1) \\ & = \int_0^n (u^2+1)^{\frac{\mu(\kappa_0+\kappa_2)}{\delta^2}} du + \frac{\delta^2}{\kappa_0+\kappa_2}\left[\frac{\mu(\kappa_0+\kappa_2)(1+\kappa_0+\kappa_2)}{\delta^2+\mu(\kappa_0+\kappa_2)}\right]^{\frac{\mu(\kappa_0+\kappa_2)}{\delta^2}+1} \to \infty \text{ as } n \to \infty. \end{cases} \tag{4.33}$$

To show that $\tilde{y_1}(t)$ is regular, we define a function

$$W_1(t, y_1) = V_1(t, y_1)\exp\{-(t - t_0)\}, \tag{4.34}$$

From (4.32), we note that $\mathbf{L}W_1 \leq 0$. By defining $\tau_1^{(n)}(t) = \min(\tau_1^{(n)}, t)$ and imitating the argument of Lemma 3.2 of [57], we have

$$\mathbb{E}\{V_1(\tau_1^{(n)}(t), \tilde{y}_1(\tau_1^{(n)}(t))) \leq e^{(t-t_0)}\mathbb{E}V_1(t_0, y_1(t_0)).$$

Hence

$$\mathcal{P}\{\tau_1^{(n)} \leq t\} \leq \frac{e^{(t-t_0)}\mathbb{E}V_1(t_0, y_1(t_0))}{\inf\limits_{|y_1|>n, u>t_0} V_1(u, y_1)} \to 0 \text{ as } n \to \infty \text{ by } (4.33). \tag{4.35}$$

Thus, using (4.26) and (4.35), the global existence and uniqueness follows by letting $n \to \infty$.

From (4.29), we conclude the global existence of $y_1(t)$ of (4.14) which is an almost surely unique continuous stochastic process. Hence, using (4.12)(a), we can also show that there exist the global solution $x_1(t)$ of sub-system of (4.8) which is an almost surely continuous and unique stochastic process .

For the proof of the global existence of solution $y_2(t)$ of the second component of (4.14), we show the existence of solution $\sigma^2(t, x_{2t})$ of the third component. The existence and uniqueness of $\sigma^2(t, x_{2t})$ follows from Theorem $A.1$ in Appendix $A.1$.

For the proof of the existence of y_2, we note that from the boundedness of functional $\sigma(t, \vartheta_2)$ and the minimal class of functions [72], we have

$$||\sigma(t, \vartheta_2)|| \leq \eta\sqrt{|\vartheta_2(0)|}, \tag{4.36}$$

for some positive constant $\eta > 0$. From the proof of global existence and almost sure stability of first component of (4.14), we assume that

$$|y_1(t)| \leq M \ \ \forall \ \ t \geq t_0 \tag{4.37}$$

for some positive constant M. For the proof of the global existence of y_2, we define a non-negative Lyapunov function

$$V_2(t, y_2) = \int_0^{y_2} (u^2+1)^{\frac{\gamma}{2\eta^2}} du + \max_{y_2 \in [0, 2(M+\kappa_1+\kappa_2)]} \left(\gamma (y_2^2+1)^{\frac{\gamma}{2\eta^2}} \left[(M+\kappa_1+\kappa_2)|y_2| - \frac{1}{2} y_2^2 \right] \right). \tag{4.38}$$

The $\mathbf{L}$-operator with respect to the second component of (4.14) is given by

$$\begin{aligned}
\mathbf{L}V_2 &= \gamma y_2(y_1 - y_2 + \kappa_1 + \kappa_2)(y_2^2+1)^{\frac{\gamma}{2\eta^2}} + \gamma \frac{\sigma^2(t, y_{2t} - \kappa_1)}{2\eta^2} (y_2^2+1)^{\frac{\gamma}{2\eta^2}-1} y_2^3 \\
&\leq \gamma y_2(y_1 - y_2 + \kappa_1 + \kappa_2)(y_2^2+1)^{\frac{\gamma}{2\eta^2}} + \frac{\gamma}{2}(y_2^2+1)^{\frac{\gamma}{2\eta^2}} y_2^2 \\
&= \gamma (y_2^2+1)^{\frac{\gamma}{2\eta^2}} \left[-\frac{1}{2} y_2^2 + y_2(y_1 + \kappa_1 + \kappa_2) \right].
\end{aligned}$$

Case 1: If $-\frac{1}{2}y_2^2 + y_2(y_1+\kappa_1+\kappa_2) < 0$, then $LV_2 < 0 \leq V_2$.

Case 2: If $-\frac{1}{2}y_2^2 + y_2(y_1+\kappa_1+\kappa_2) \geq 0$, then $0 \leq |y_2| \leq 2|y_1+\kappa_1+\kappa_2|$.

Since continuous functions on closed intervals are bounded, then the function $f(y_2) = \gamma(y_2^2 + 1)^{\frac{\gamma}{2\eta^2}} \left[-\frac{1}{2}y_2^2 + y_2(y_1+\kappa_1+\kappa_2)\right]$ is bounded on the interval $0 \leq y_2 \leq 2(M+\kappa_1+\kappa_2)$. Hence, for $y_2 \in [0, 2(M+\kappa_1+\kappa_2)]$,

$$\begin{aligned}
\mathbf{L}V_2 &\leq \gamma (y_2^2+1)^{\frac{\gamma}{2\eta^2}} \left[-\frac{1}{2} y_2^2 + y_2(y_1 + \kappa_1 + \kappa_2) \right] \\
&\leq \gamma (y_2^2+1)^{\frac{\gamma}{2\eta^2}} \left[-\frac{1}{2} y_2^2 + |y_2|(M + \kappa_1 + \kappa_2) \right] \\
&\leq V_2.
\end{aligned}$$

Furthermore,

$$\begin{cases} V_{2_n} &= \inf\limits_{|y_2|>n} V_2(t,y_2) \\ &= \int_0^n (u^2+1)^{\frac{\gamma}{2\eta^2}}du + \max\limits_{y_2\in[0,2(M+\kappa_1+\kappa_2)]}\left(\gamma(y_2^2+1)^{\frac{\gamma}{2\eta^2}}\left[(M+\kappa_1+\kappa_2)y_2-\frac{1}{2}y_2^2\right]\right). \end{cases} \tag{4.39}$$

It follows that $V_{2_n}\to\infty$ as $n\to\infty$. By defining

$$W_2(t,y_2)=V_2(t,y_2)e^{-(t-t_0)}, \tag{4.40}$$

we have

$$\mathcal{P}\{\tau_2^{(n)}\le t\}\le\frac{e^{(t-t_0)}\mathbb{E}V_2(t_0,y_2(t_0))}{\inf\limits_{|y_2|>n,u>t_0}V_2(u,y_2)}\to 0 \text{ as } n\to\infty \text{ by } (4.39). \tag{4.41}$$

Thus, the global existence and uniqueness of solution of the second component of (4.14) follows by letting $n\to\infty$. Hence, there exist a global solution $(y_1(t),y_2(t))$ of the system of non-linear stochastic equation (4.14). □

Using transformation (4.12), it can be easily deduced that there exist a global solution $(x_1(t),x_2(t))$ of the system of non-linear stochastic system (4.8).

4.4 Closed Form Solution Under $\mathcal{P}$

We observe that the system of stochastic non-linear differential equations (4.8) is a Itô-Doob stochastic Bernoulli type stochastic differential equations [70]

$$dy=\left[P(t)y+Q(t)y^n+\frac{n}{2}\Upsilon^2(t)y^{2n-1}\right]dt+\left[\Sigma(t)y+\Upsilon(t)y^n\right]dW_(t) \tag{4.42}$$

for any $n\neq 1$, where P,Q,Σ and Υ are continuous functions.

To find solutions $y_1(t)$ and $y_2(t)$, we imitate the procedure [70] for finding the implicit-closed form solution processes of first two components of non-linear stochastic differential equations in (4.14).

We consider an Energy/Lyapunov function

$$\mathbb{V}_i(t,y_i)=\frac{1}{y_i(t)}, \text{ for } i=1,2, \ y_i(t)\neq 0, \tag{4.43}$$

Hence, applying It$\hat{o}$'s formula to (4.43), we have

$$\begin{cases} d\mathbb{V}_1 &= \left[(\mu\lambda_1+\delta^2)\mathbb{V}_1+\mu\right]dt+\delta\mathbb{V}_1 dW_1(t) \\ d\mathbb{V}_2 &= \left[(-\gamma(\lambda_2+y_1(t))+\sigma^2(t,\varphi_2))\mathbb{V}_2+\gamma\right]dt-\sigma(t,\varphi_2-\kappa_1)\mathbb{V}_2 dW_2(t). \end{cases} \tag{4.44}$$

Using the techniques described in [70], the implicit solution to system of differential equation (4.44) is given by

$$\begin{cases} \mathbb{V}_1(t,y_1) &= \phi_1(t,t_0)c_1+\mu\int_{t_0}^t \phi_1(t,s)ds \\ \mathbb{V}_2(t,y_2) &= \phi_2(t,t_0)c_2+\gamma\int_{t_0}^t \phi_2(t,s)ds \end{cases} \tag{4.45}$$

where

$$\begin{cases} \phi_1(t,t_0) &= \exp\left[\left(\mu(\kappa_2+\kappa_0)+\frac{1}{2}\delta^2\right)(t-t_0)+\delta\left(W_1(t)-W_1(t_0)\right)\right] \\ \phi_2(t,t_0) &= \exp\left[\int_{t_0}^t\left(-\gamma(y_1(s)+\lambda_2)+\frac{1}{2}\sigma^2(s,y_{2s}-\kappa_1)\right)ds-\int_{t_0}^t\sigma(s,y_{2s}-\kappa_1)dW_2(s)\right], \end{cases} \tag{4.46}$$

and c_i, $i=1,2$ are constants.

Comparing (4.43) and (4.45), we have

$$\begin{cases} y_1(t)=\left[\phi_1(t,t_0)c_1+\mu\int_{t_0}^t\phi_1(t,s)ds\right]^{-1} \\ y_2(t)=\left[\phi_2(t,t_0)c_2+\gamma\int_{t_0}^t\phi_2(t,s)ds\right]^{-1}. \end{cases} \tag{4.47}$$

Hence, using transformation (4.12) together with the initial condition $y_1(t_0)=y_{10}>0$, $y_{2t_0}=\varphi_{02}>0$, we have

$$\begin{cases} x_1(t)=\left[\frac{\phi_1(t,t_0)}{x_{10}-\kappa_2}+\mu\int_{t_0}^t\phi_1(t,s)ds\right]^{-1}+\kappa_2 \\ x_2(t)=\left[\frac{\phi_2(t,t_0)}{\vartheta_{02}+\kappa_1}+\gamma\int_{t_0}^t\phi_2(t,s)ds\right]^{-1}-\kappa_1. \end{cases} \tag{4.48}$$

REMARK 6 It is obvious from (4.47) that $y_i>0$ for $i=1,2$. Also, $\phi_1(t,t_0)$ is a log-normal random variable $\log\mathcal{N}\left(\left[(\mu(\kappa_0+\kappa_2)+\frac{1}{2}\delta^2)\right](t-t_0),\delta^2(t-t_0)\right)$. Hence

$$\mathbb{E}_{\mathcal{P}}\phi_1(t,t_0)=\exp\left[(\mu(\kappa_0+\kappa_2)+\delta^2)(t-t_0)\right]. \tag{4.49}$$

By Jensen's inequality, we have

$$\begin{aligned} \mathbb{E}_{\mathcal{P}}\left[|y_1(t)|\right] &\geq \left[\mathbb{E}_{\mathcal{P}}\left(\frac{\phi_1(t,t_0)}{y_{10}}+\mu\int_{t_0}^t\phi_1(t,s)ds\right)\right]^{-1} \\ &= \left[\left(\frac{1}{y_{10}}+\frac{\mu}{\mu(\kappa_0+\kappa_2)+\delta^2}\right)\exp\left[(\mu(\kappa_0+\kappa_2)+\delta^2)(t-t_0)\right]\right. \\ &\quad \left.-\frac{\mu}{\mu(\kappa_0+\kappa_2)+\delta^2}\right]^{-1}. \end{aligned} \tag{4.50}$$

Hence,

$$\lim_{t\to\infty}\mathbb{E}_{\mathcal{P}}\left[|y_1(t)|\right]\geq 0 \tag{4.51}$$

Also, since

$$y_1(t) \leq \left[\frac{\phi_1(t,t_0)}{y_{10}}\right]^{-1}, \tag{4.52}$$

we have

$$\mathbb{E}_{\mathcal{P}}\left[y_1(t)\right] \leq y_{10}\exp(-\mu(\kappa_0+\kappa_2)(t-t_0)). \tag{4.53}$$

Hence, by Squeeze theorem, from (4.50) and (4.53),

$$\lim_{t\to\infty}\mathbb{E}_{\mathcal{P}}\left[y_1(t)\right] = 0. \tag{4.54}$$

Consequently, using (4.12), we have

$$\lim_{t\to\infty}\mathbb{E}_{\mathcal{P}}\left[x_1(t)\right] = \kappa_2. \tag{4.55}$$

This establishes the fact that $x_1(t)$ describes the mean of non-seasonal log-spot price.

We can also evaluate the area under the curve $y_i(t)$ from t_0 to t. To do this, we re-write (4.47) as

$$\begin{cases} y_1(t) = \frac{\phi_1^{-1}(t,t_0)y_{01}}{1+\mu y_{01}\int_{t_0}^t \phi_1^{-1}(s,t_0)ds} \\ y_2(t) = \frac{\phi_2^{-1}(t,t_0)\varphi_{02}}{1+\gamma\varphi_{02}\int_{t_0}^t \phi_2^{-1}(s,t_0)ds}. \end{cases} \tag{4.56}$$

It follows immediately that

$$\int_{t_0}^t y_1(s)ds = \frac{1}{\mu}\ln\left[1+\mu y_{01}\int_{t_0}^t \phi_1^{-1}(s,t_0)ds\right], \tag{4.57}$$

$$\int_{t_0}^t y_2(s)ds = \frac{1}{\gamma}\ln\left[1+\gamma\varphi_{02}\int_{t_0}^t \phi_2^{-1}(s,t_0)ds\right]. \tag{4.58}$$

Hence, applying Fubini's theorem, using the concavity of logarithmic function, and the facts that

$$\begin{aligned} &\mathbb{E}_{\mathcal{P}}\phi_1^{-1}(t,t_0) = \exp\left[-\mu(\kappa_0+\kappa_2)(t-t_0)\right], \\ &\mathbb{E}_{\mathcal{P}}\phi_2^{-1}(t,t_0) = \exp\left[\gamma\int_{t_0}^t(\mathbb{E}_{\mathcal{P}}[y_1(s)]+\lambda_2)ds\right], \end{aligned} \tag{4.59}$$

and $\mathbb{E}_{\mathcal{P}}[y_1(t)] \leq y_{10}$ (from (4.53)), we have

$$\begin{aligned} \int_{t_0}^t \mathbb{E}_{\mathcal{P}}[y_1(s)]ds &\leq \frac{1}{\mu}\ln\left[1+\mu\mathbb{E}_{\mathcal{P}}[y_{01}]\int_{t_0}^t \mathbb{E}_{\mathcal{P}}[\phi_1^{-1}(s,t_0)]ds\right], \\ &\leq \frac{1}{\mu}\ln\left[1+\frac{\mathbb{E}_{\mathcal{P}}[y_{10}]}{\kappa_0+\kappa_2}\left(1-e^{-\mu(\kappa_0+\kappa_2)(t-t_0)}\right)\right], \\ \int_{t_0}^t \mathbb{E}_{\mathcal{P}}[y_2(s)]ds &\leq \frac{1}{\gamma}\ln\left[1+\gamma\mathbb{E}_{\mathcal{P}}[\varphi_{02}]\int_{t_0}^t \mathbb{E}_{\mathcal{P}}[\phi_2^{-1}(s,t_0)]ds\right], \\ &\leq \frac{1}{\gamma}\ln\left[1+\frac{\mathbb{E}_{\mathcal{P}}[\varphi_{02}]}{y_{10}+\lambda_2}\left(e^{\gamma(y_{10}+\lambda_2)(t-t_0)}-1\right)\right], \end{aligned} \tag{4.60}$$

In addition to the above outlined results, we present a few more properties of y_1 and y_2.

THEOREM 4.2 *If* $\gamma - \frac{\eta^2}{2} > 0$*, then*

$$\mathbb{E}_{\mathcal{P}}[y_2(t)] \leq \left[\frac{2\gamma - \eta^2}{2\gamma(\kappa_1 + \kappa_2 + M)} + \left(f_0 - \frac{2\gamma - \eta^2}{2\gamma(\kappa_1 + \kappa_2 + M)}\right) e^{-\gamma(\kappa_1 + \kappa_2 + M)(t - t_0)}\right]^{-1}, \quad (4.61)$$

where M *is defined in* (4.37)*, where* $f_0 = \frac{1}{\sqrt{\mathbb{E}[y_2^2(t)]}}|_{t=t_0}$.

Proof. Using the fact that $y_2 > 0$, define the Lyapunov function $v : [t_0, \infty) \times \mathbb{R}_+ \to \mathbb{R}^+$ by

$$v(t, y_2) = y_2^2. \quad (4.62)$$

Then from (4.14)

$$dv = \mathbf{L}v dt + 2\sigma(t, y_{2t}) v dW_2(t), \quad (4.63)$$

where the operator $\mathbf{L}$ is define as

$$\mathbf{L} = \frac{\partial}{\partial t} + \gamma y_2(\kappa_1 + \kappa_2 + y_1 - y_2)\frac{\partial}{\partial y_2} + \frac{1}{2}\sigma^2(t, y_{2t}) y_2^2 \frac{\partial^2}{\partial y_2^2}. \quad (4.64)$$

Using (4.36) and (4.37), the operator $\mathbf{L}$ satisfies

$$\mathbf{L}v \leq -(2\gamma - \eta^2) v^{\frac{3}{2}} + 2\gamma(\kappa_1 + \kappa_2 + M) v. \quad (4.65)$$

Define $u(t) = \mathbb{E}_{\mathcal{P}}(v(t, y_2(t)))$. By applying Theorem 4.8.1 of [66], we obtain

$$\mathbb{E}[v(t, y_2(t))] \leq u(t, t_0, u_0), \quad (4.66)$$

where $u(t, t_0, u_0)$ is a solution of

$$du(t) = \left[-(2\gamma - \eta^2) u^{\frac{3}{2}}(t) + 2\gamma(\kappa_1 + \kappa_2 + M) u(t)\right] dt, \quad (4.67)$$

Thus,

$$\mathbb{E}_{\mathcal{P}}(y_2^2(t)) \leq \left[\frac{2\gamma - \eta^2}{2\gamma(\kappa_1 + \kappa_2 + M)} + \left(f_0 - \frac{2\gamma - \eta^2}{2\gamma(\kappa_1 + \kappa_2 + M)}\right) e^{-\gamma(\kappa_1 + \kappa_2 + M)(t - t_0)}\right]^{-2}. \quad (4.68)$$

By using Hölder's inequality, inequality (4.61) follows .

□

THEOREM 4.3 *If* $4\gamma(\kappa_1 + \kappa_2) > (2\gamma + 3\eta^2)$*, then*

$$\mathbb{E}_{\mathcal{P}}[|y_2(t)|] \geq \frac{1}{\sqrt{\left(\frac{1}{\varphi_{02}^2} - \frac{\beta_1}{\alpha_1}\right) e^{-\alpha_1(t - t_0)} + \frac{\beta_1}{\alpha_1}}}, \quad (4.69)$$

where

$$\begin{cases} \beta_1 &= \gamma + \frac{3}{2}\eta^2, \\ \alpha_1 &= 2\gamma(\kappa_1+\kappa_2) - \beta_1. \end{cases} \tag{4.70}$$

Proof. Using the fact that $y_2 > 0$, define the Lyapunov function $\mathbf{v} : [t_0,\infty) \times \mathbb{R}_+ \to \mathbb{R}^+$ by

$$\mathbf{v}(t, y_2) = \frac{1}{y_2^2}. \tag{4.71}$$

Then from (4.14) and (4.36), using the fact that $\frac{2}{y_2} \le \frac{1}{y_2^2} + 1$, we have

$$\begin{aligned} d\mathbf{v} &= \left[-\frac{2}{y_2^3}[\gamma y_2(\kappa_1+\kappa_2+y_1-y_2)] + \frac{3}{y_2^2}\sigma^2(t, y_{2t}-\kappa_1)\right]dt - \frac{2}{y_2^2}\sigma(t, y_{2t}-\kappa_1)dW_2(t), \\ &\le [-\alpha_1\mathbf{v} + \beta_1]dt - 2\mathbf{v}\sigma(t, y_{2t}-\kappa_1)dW_2(t). \end{aligned}$$

Thus,

$$d(\mathbf{v}e^{\alpha_1 t}) \le \beta_1 e^{\alpha_1 t}dt - 2\mathbf{v}\sigma(t, y_{2t}-\kappa_1)dW_2(t).$$

Hence,

$$\mathbb{E}_{\mathcal{P}}[\mathbf{v}(t)] \le \left(\mathbf{v}_0 - \frac{\beta_1}{\alpha_1}\right)e^{-\alpha_1(t-t_0)} + \frac{\beta_1}{\alpha_1}.$$

Applying Jensen's inequality, the result follows. □

4.5 Risk-Neutral Dynamics

In this section, we present a risk-neutral dynamic model corresponding to (4.8).

DEFINITION 4.5.1 *A probability measure $\bar{\mathcal{P}}$ is said to be risk-neutral if*

- *$\bar{\mathcal{P}}$ and $\mathcal{P}$ are equivalent (that is, for every $A \in \mathcal{F}$, $\mathcal{P}(A) = 0$ if and only if $\bar{\mathbb{P}}(A) = 0$), and*

- *Under $\bar{\mathcal{P}}$, the discounted price $D(t)$ is a martingale.*

We shall use this definition to find a risk-neutral dynamics for our model (4.8).

Define the riskless asset

$$B_i(t) = \exp\left[\int_{t_0}^{t} r_i(s)ds\right], \quad t \in [0,T], \quad i = 1,2 \tag{4.72}$$

where r_i, $i = 1,2$ are the interest rate function.

Using the first two components of (4.14), define the discounted price of $y_1(t) = x_1(t) - \kappa_2$, $y_2(t) = x_2(t) + \kappa_1$, by

$$D_i(t) = \frac{\varphi_i(0)}{B_i(t)} = \exp\left[-\int_{t_0}^{t} r_i(s)ds\right] y_i(t). \tag{4.73}$$

Applying Itô's Lemma to (4.73), we have

$$\begin{cases} dD_1 &= -\delta D_1 \left[\frac{\mu(\kappa_0+\kappa_2+y_1)+r_1}{\delta}dt + dW_1(t)\right] \\ dD_2 &= \sigma(t, y_{2t} - \kappa_1) D_2 \left[\frac{\gamma(y_1+\kappa_1+\kappa_2-y_2)-r_2}{\sigma(t,y_{2t}-\kappa_1)}dt + dW_2(t)\right]. \end{cases} \tag{4.74}$$

Define the market price of risk

$$\begin{cases} \theta_1 &= \frac{\mu(\kappa_0+\kappa_2+y_1)+r_1}{\delta} \\ \theta_2 &= \frac{\gamma(y_1+\kappa_1+\kappa_2-y_2)-r_2}{\sigma(t,\varphi_2-\kappa_1)}, \end{cases} \tag{4.75}$$

where φ_i, $i = 1, 2$ are as defined in (4.15).

Using Girsanov's theorem, we obtain the following result concerning the change of probability measure.

THEOREM 4.4 *Suppose that θ_i, $i = 1, 2$ satisfy the Novikov's condition [108], with the $\bar{\mathcal{P}}$-Wiener process*

$$\begin{cases} \bar{W}_1(t) &= W_1(t) + \int_{t_0}^{t} \theta_1(u)du, \\ \bar{W}_2(t) &= W_2(t) + \int_{t_0}^{t} \theta_2(u)du. \end{cases} \tag{4.76}$$

Then $D_i(t)$ is a positive local martingale with respect to $\bar{\mathcal{P}}$, and is given by

$$\begin{cases} D_1(t) &= D_{10} \exp\left[-\frac{1}{2}\int_{t_0}^{t} \delta^2 ds - \int_{t_0}^{t} \delta d\bar{W}_1(s)\right] \\ D_2(t) &= D_{20} \exp\left[-\frac{1}{2}\int_{t_0}^{t} \sigma^2(s, y_{2s} - \kappa_1)ds + \int_{t_0}^{t} \sigma(s, y_{2s} - \kappa_1)d\bar{W}_2(s)\right]. \end{cases} \tag{4.77}$$

Substituting (4.76) into (4.14), we notice that first two component of (4.14) reduces to a geometric stochastic equation given by

$$\begin{cases} dy_1 &= r_1(t)y_1 dt - \delta y_1 d\bar{W}_1 \\ dy_2 &= r_2(t)y_2 dt + \sigma(t, y_{2s} - \kappa_1)y_2 d\bar{W}_2 \end{cases} \tag{4.78}$$

Using transformation (4.12), (4.78) reduces to

$$\begin{cases} dx_1 &= -r_1(\kappa_2 - x_1)dt + \delta(\kappa_2 - x_1)d\bar{W}_1 \\ dx_2 &= r_2(x_2 + \kappa_1)dt + \sigma(t, x_{2t})(x_2 + \kappa_1)d\bar{W}_2 \end{cases} \tag{4.79}$$

Chapter 5

Parameter Estimation

5.1 Introduction

In this chapter, we present the estimation scheme to estimate the parameters in the interconnected system of nonlinear stochastic differential equation (4.11). We use discretized Scheme for Continuous-Time GARCH Model to develop the Maximum Likelihood techniques. the developed techniques is used to estimate parameters in the model for volatility process in (4.11). Furthermore, modifying the extended Kalman filter technique, we estimate the parameters in the model for log-spot and expected log-spot price in (4.11).

The Kalman Filter is a powerful and widely used technique in state and parameter estimation problems. It is used for finding minimum mean squared error (MMSE) estimation of linear state dynamic systems and observations [115]. Nonlinear state dynamic and observations are estimated by employing the Extended Kalman Filter (EKF) scheme [115]. Moreover, the EKF scheme deals with state and parameter estimation of linearized version of both nonlinear state dynamic and observations [73]. It is well known [78] that the linearized Taylor scheme does not provide sufficiently accurate representation. Moreover, due to its overly crude approximation, the scheme generates problem in convergence [78].

Several other approaches have been made to find a better filter than the EKF scheme. Unlike the usual EKF approach, Magnus [78], Tor Steinar [124] and Luo [75] propose a new set of estimators which are based on polynomial approximations of the nonlinear transformations using the Stirling's interpolation formula. Under this scheme, derivatives of rate functions are avoided due to interpolation approximation formula. As discussed in [78], the Stirling's interpolation formula accommodates easy implementation of the filters and enables state estimation when the derivatives are not smooth. It has been remarked that this approach provides a similar, or superior performance than the existing EKF approach. Simon Julier [113, 114, 115] claims that the EKF filtering strategy is difficult to implement, difficult to tune, and only reliable for systems which are almost linear. This leads to the development of a new linear state and covariance estimator using unscented transformation. The new scheme was claimed to be superior than that of the EKF, and, in fact, the scheme generalizes elegantly to the nonlinear system without the linear step required by the EKF scheme. Higher filters have also been discussed by Jazwinski, [53], Maybeck, [81], and Madsen etal [76].

Our main focus in this paper is to reduce the magnitude of error that occurs during the estimation process of the EKF approach. This error is due to the overly simplified approximation scheme. In the process of the error reduction, we modified the Extended Kalman Filter scheme by incorporating second order polynomial approximation for the expected state variable and covariance. This scheme is applied to study the state and parameter estimation problems of nonlinear system of stochastic differential equation. The drift and diffusion part of the nonlinear differential equations are approximated using the Stirling's interpolation formula [78]. This modified approach estimates the parameters of a system of nonlinear stochastic differential equation with lesser magnitude of error compared to the usual EKF approach [73]. Although the magnitude of error in the state and covariance of the EKF is reduced, it is however important to note that our scheme is computationally too demanding/computer intensive. An algorithm is developed to implement this scheme. The extended Kalman filter approach is compared with the developed modified extended Kalman filtering approach. The scheme is applied to Henry Hub natural gas data and to estimate parameters. The details are exhibited in the graph.

The organization of this work is as follows:

In Section 5.2, we present the discretized scheme for continuous-time GARCH Model. In Section 5.3, we present a modified EKF scheme. In Section 5.4, we applied the scheme to estimate the parameters for a stochastic dynamic model for Henry Hub Natural gas.

5.2 Discretized Scheme for Continuous-Time GARCH Model (4.10)

In this section, we formulate a discretized scheme and outline a procedure for estimating the parameters α, β, τ and c in (4.10). An outline of the procedure is given below:

Define the discrete-time delay value l to be the analogue of the continuous-time delay τ. Given the value of l, we define the size of the mesh of the discrete-time grid as $\Delta = \frac{\tau}{l}$. Furthermore, we define

$$\varepsilon_i = \sigma_i \xi_i, \tag{5.1}$$

where ξ_i is a white noise process. The discrete-time delayed model corresponding to (4.10) for volatility is described by

$$\sigma_n^2 = \alpha + \beta\Delta t\left[\sum_{i=1}^{l}\varepsilon_{n-i}\right]^2 + q\sigma_{n-1}^2. \tag{5.2}$$

$\sigma_i^2 = \sigma_0^2$ for $i \in [-\tau, o]$, and $q = 1 + c$.

Since ε_j is a normal random variable with mean 0 and variance σ_j^2, we can write

$$\sum_{i=1}^{l}\varepsilon_{n-i} \equiv \sqrt{\sum_{i=1}^{l}\sigma_{n-i}^2}\ \epsilon, \tag{5.3}$$

where ϵ is a standard normal variable. Hence, (6.1) reduces to

$$\sigma_n^2 = \alpha + \beta\Delta t\sum_{i=1}^{l}\sigma_{n-i}^2\epsilon^2 + q\sigma_{n-1}^2, \tag{5.4}$$

Using the fact that ϵ^2 is a $\chi^2(1)$ random variable, we find the probability density function $f(\sigma_n^2|\sigma_{n-i}^2, 1 \le i \le l)$ of σ_n^2 given $\sigma_{n-i}^2, 1 \le i \le l$ to be

$$f(\sigma_n^2|\sigma_{n-i}^2, 1 \le i \le l) = \frac{(\sigma_n^2 - \alpha - q\sigma_{n-1}^2)^{-\frac{1}{2}}}{\sqrt{2\pi\beta\Delta t\sum\limits_{i=1}^{l}\sigma_{n-i}^2}}\exp\left[-\frac{\sigma_n^2 - \alpha - q\sigma_{n-1}^2}{\beta\Delta t\sum\limits_{i=1}^{l}\sigma_{n-i}^2}\right], \alpha+q\sigma_{n-1}^2 < \sigma_n^2 < \infty. \tag{5.5}$$

We define the Likelihood function of σ_n^2 as

$$\mathcal{L}(\text{II}_3) = \log\prod_{n=1}^{N} f(\sigma_n^2|\text{II}_3, \sigma_{n-i}^2, 1 \le i \le l) \tag{5.6}$$

where $\text{II}_3 = \{\alpha, \beta, q\}$ are the parameters to be estimated. Thus,

$$\mathcal{L}(\text{II}_3) = -\frac{1}{2}\sum_{n=1}^{N}\ln\left[\frac{\sigma_n^2 - \alpha - q\sigma_{n-1}^2}{2\pi\beta\Delta t\sum\limits_{i=1}^{l}\sigma_{n-i}^2}\right] - \sum_{n=1}^{N}\left[\frac{\sigma_n^2 - \alpha - q\sigma_{n-1}^2}{\beta\Delta t\sum\limits_{i=1}^{l}\sigma_{n-i}^2}\right]. \tag{5.7}$$

Our aim is to find estimators that maximize the Likelihood function (5.7) subject to the constraint (2.49).

Hence, solving for the maximum-likelihood estimators $\hat{\alpha}, \hat{\beta}$ and $\hat{q}$ of α, β and q respectively, we have

$$\hat{\beta} = \frac{2}{N}\sum_{n=1}^{N}\left[\frac{\sigma_n^2 - \hat{\alpha} - \hat{q}\sigma_{n-1}^2}{\Delta t\sum\limits_{i=1}^{l}\sigma_{n-i}^2}\right],$$

and $\hat{\alpha}$, $\hat{q}$ satisfies

$$\begin{aligned}\frac{1}{2}\sum_{n=1}^{N}\left[\frac{1}{\sigma_n^2-\hat{\alpha}-\hat{q}\sigma_{n-1}^2}\right]+\sum_{n=1}^{N}\left[\frac{1}{\hat{\beta}\Delta t\sum\limits_{i=1}^{l}\sigma_{n-i}^2}\right] &= 0,\\ \frac{1}{2}\sum_{n=1}^{N}\left[\frac{\sigma_{n-1}^2}{\sigma_n^2-\hat{\alpha}-\hat{q}\sigma_{n-1}^2}\right]+\sum_{n=1}^{N}\left[\frac{\sigma_{n-1}^2}{\hat{\beta}\Delta t\sum\limits_{i=1}^{l}\sigma_{n-i}^2}\right] &= 0,\end{aligned}$$

respectively.

To evaluate the parameters, we generate the observation data for $\sigma(t,\vartheta_2)$ from the discrete version of (4.14) described as

$$\frac{\Delta y_2}{y_2}=\gamma(\kappa_1+\kappa_2+y_1-y_2)\Delta t+\sigma(t,y_{2t})\Delta W_2. \tag{5.8}$$

We achieve this by using y_2 as our observation data. We search iteratively to find the parameters that maximize (5.7) using a combination of direct search method and the Nelder-Mead simplex optimization algorithm in Matlab. This completes the parameter estimation problem of (4.10). The parameter estimated are recorded in Table 3.

Table 3: Estimated Parameters of $\sigma^2(t,\vartheta_2)$ for $l=2$ using the Henry Hub daily natural gas spot prices for the period 01/04/2000-09/30/2004 [24].

α	β	c	τ
0.07	1.149	-1.4814	0.005

5.3 Modified Extended Kalman Filter Approach

In this section, we shall be estimating the remaining parameters $\mu,\gamma,\kappa_0,\kappa_1,\kappa_2$ and δ of (4.8) using the Modified Extended modified Kalman Filter Approach. This is accomplished by approximating the state estimator using a quadratic approximation. The Kalman Filter Approach is modified by employing a second order approximation for state and state variance predictions. To estimate the parameters, we minimized the likelihood function of the prediction error of the measurement process. The approach is described below.

We assume that a dynamic state $x \in \mathbb{R}^n$ and its observation data $y \in \mathbb{R}^n$ are described by a general non-linear stochastic dynamic systems.

$$\begin{cases} dx &= \mathbf{f}(x;\boldsymbol{\theta})dt + \mathbf{g}(x;\boldsymbol{\theta})d\mathbf{W}(t), \quad x(t_0) = x_0 \\ y(t) &= \mathbf{h}(x;\boldsymbol{\theta}) + \mathbf{v}(t), \end{cases} \tag{5.9}$$

where x_0 is a stochastic initial condition satisfying $E|\mathbf{x}_0|^2 < \infty$, $\mathbf{f} : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$, $\mathbf{g} : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^{n\times d}$, $\mathbf{h} : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ are continuous functions, $\mathbf{W} : \mathbb{R} \to \mathbb{R}^d$ is a $d-$ dimensional standard Wiener process on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{t\geq 0}, \mathcal{P})$, the filtration function $(\mathcal{F})_{t\geq 0}$ is right-continuous, and each $\mathcal{F}_t$ with $t \geq 0$ contains all $\mathcal{P}$-null sets in $\mathcal{F}$, x is $\mathcal{F}_t$ adapted process and non-anticipative, and $v : \mathbb{R} \to \mathbb{R}^n$ is a $n-$ dimensional zero mean Gaussian white noise process independent of $\mathbf{W}$, $\theta \in \Theta$, the parameter space.

Prior to presenting a procedure for the estimation of parameters, we define the following terminologies and notations used throughout this work.

Define

$$Y_{t_k} = \{y_{t_1}, y_{t_2}, ..., y_{t_k}\}, \tag{5.10}$$

as all observations of the data given up to time t_k.

$$\begin{aligned}
\hat{y}(t|t_{k-1}) &= \mathbb{E}\left[y(t)|Y_{t_{k-1}}\right], \\
\hat{x}(t|t_{k-1}) &= \mathbb{E}\left[x(t)|Y_{t_{k-1}}\right], \\
P(t|t_{k-1}) &= \mathbb{E}\left[(x(t)-\hat{x}(t|t_{k-1}))(x(t)-\hat{x}(t|t_{k-1}))^T|Y_{t_{k-1}}\right], \\
R(t|t_{k-1}) &= \mathbb{E}\left[v(t)v^T(t)|Y_{t_{k-1}}\right], \\
r_{0,2}(t|t_{k-1}) &= \mathbb{E}\left[(y(t)-\hat{y}(t|t_{k-1}))(y(t)-\hat{y}(t|t_{k-1}))^T|Y_{t_{k-1}}\right], \\
r_{1,1}(t|t_{k-1}) &= \mathbb{E}\left[(x(t)-\hat{x}(t|t_{k-1}))(y(t)-\hat{y}(t|t_{k-1}))^T|Y_{t_{k-1}}\right], \\
r_{2,2}(t|t_{k-1}) &= \mathbb{E}\Big[(x(t)-\hat{x}(t|t_{k-1}))\,(x(t)-\hat{x}(t|t_{k-1}))^T\,(y(t)-\hat{y}(t|t_{k-1}))\times \\
&\quad (y(t)-\hat{y}(t|t_{k-1}))^T|Y_{k-1}\Big], \\
r_{1,2}(t|t_{k-1}) &= \mathbb{E}\left[x(t)(y(t)-\hat{y}(t|t_{k-1}))^T(\mathbb{Y}(t)-\hat{\mathbb{Y}}(t|t_{k-1}))^T|Y_{t_{k-1}}\right], \\
r_{0,3}(t|t_{k-1}) &= \mathbb{E}\left[(y(t)-\hat{y}(t|t_{k-1}))(y(t)-\hat{y}(t|t_{k-1}))^T(\mathbb{Y}(t)-\hat{\mathbb{Y}}(t|t_{k-1}))^T|Y_{t_{k-1}}\right], \\
r_{1,3}(t|t_{k-1}) &= \mathbb{E}\Big[(x(t)-\hat{x}(t|t_{k-1}))(y(t)-\hat{y}(t|t_{k-1}))^T(y(t)-\hat{y}(t|t_{k-1}))\times \\
&\quad (y(t)-\hat{y}(t|t_{k-1}))^T|Y_{t_{k-1}}\Big]
\end{aligned} \tag{5.11}$$

$$
\begin{aligned}
r_{0,4}(t|t_{k-1}) &= \mathbb{E}\left[(y(t)-\hat{y}(t|t_{k-1}))(y(t)-\hat{y}(t|t_{k-1}))^T(y(t)-\hat{y}(t|t_{k-1}))\times\right. \\
&\quad \left.(y(t)-\hat{y}(t|t_{k-1}))^T|Y_{t_{k-1}}\right], \\
M_{0,2}(t|t_{k-1}) &= \mathbb{E}\Big[(\mathbb{Y}(t)-\hat{\mathbb{Y}}(t|t_{k-1}))(y(t)-\hat{y}(t|t_{k-1}))(y(t)-\hat{y}(t|t_{k-1}))^T\times \\
&\quad (\mathbb{Y}(t)-\hat{\mathbb{Y}}(t|t_{k-1}))^T|Y_{t_{k-1}}\Big], \\
\sigma_{Y2}(t|t_{k-1}) &= \mathbb{E}\Big[(\mathbb{Y}(t)-\hat{\mathbb{Y}}(t|t_{k-1}))(y(t)-\hat{y}(t|t_{k-1}))|Y_{t_{k-1}}\Big],
\end{aligned}
\tag{5.12}
$$

where

$$
\mathbb{Y}(t)-\hat{\mathbb{Y}}(t|t_{k-1}) = \begin{pmatrix} \mathbf{y}(t)-\hat{\mathbf{y}}(t|t_{k-1}) & 0 & \dots & 0 \\ 0 & \mathbf{y}(t)-\hat{\mathbf{y}}(t|t_{k-1}) & \dots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & \dots & 0 & \mathbf{y}(t)-\hat{\mathbf{y}}(t|t_{k-1}) \end{pmatrix}_{n^2\times n} \tag{5.13}
$$

Let $\hat{x}(t_k|t_{k-1})$ be the a-priori state estimate at step k given the knowledge of process $Y_{t_{k-1}}$, and $\hat{x}(t_k|t_k)$ be the posterior state estimate at step k given the knowledge of process Y_{t_k}. The Extended Kalman Filter approach begins with the goal of computing a-posterior state estimate $\hat{x}(t_k|t_k)$ as a linearized approximation of the form

$$
\begin{cases} \hat{x}(t_k|t_k) &= A_0 + A_1(y(t_k)-\hat{y}(t_k|t_{k-1})), \\ P(t_k|t_k) &= B_0. \end{cases} \tag{5.14}
$$

It was shown in Jazwinski [53] that

$$
\begin{aligned}
A_0(t_k|t_{k-1}) &= \hat{x}(t_k|t_{k-1}), \\
A_1(t_k|t_{k-1}) &= r_{1,1}(t_k|t_{k-1})r_{0,2}^{-1}(t_k|t_{k-1}), \\
B_0(t_k|t_{k-1}) &= P(t_k|t_{k-1}) - A_1(t_k|t_{k-1})r_{0,2}(t_k|t_{k-1})A_1^T(t_k|t_{k-1}),
\end{aligned}
\tag{5.15}
$$

where A_1 is the Kalman gain. Instead of approximating the conditional covariance at an observation as a constant, Jazwinski [53] extended it to an approximation of order one.

For the rest of this study, for the sake of simplicity, we write $\mathbf{f}(x) = \mathbf{f}(x;\boldsymbol{\theta})$, $\mathbf{g}(x) = \mathbf{g}(x;\boldsymbol{\theta})$, and $\mathbf{h}(x) = \mathbf{h}(x;\boldsymbol{\theta})$. In this study, we extend the approximate equations for the conditional mean and covariance at an observation to that of order two. To do this, we first state the Taylors series expansion of a vector value function $\mathbf{f}$ about the vector $\hat{\mathbf{y}}$,

$$
\mathbf{f}(\mathbf{y}) = \mathbf{f}(\hat{\mathbf{y}}) + \frac{\partial \mathbf{f}(\hat{\mathbf{y}})}{\partial \mathbf{y}}(\mathbf{y}-\hat{\mathbf{y}}) + \frac{1}{2}\frac{\partial^2 \mathbf{f}(\hat{\mathbf{y}})}{\partial \mathbf{y}^2}\mathrm{diag}(\mathbf{y}-\hat{\mathbf{y}}, \mathbf{y}-\hat{\mathbf{y}}, ..., \mathbf{y}-\hat{\mathbf{y}})(\mathbf{y}-\hat{\mathbf{y}}), \tag{5.16}
$$

where $\frac{\partial \mathbf{f}(\mathbf{y})}{\partial \mathbf{y}} = \begin{pmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} & \cdots & \frac{\partial f_1}{\partial y_n} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} & \cdots & \frac{\partial f_2}{\partial y_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial y_1} & \frac{\partial f_n}{\partial y_2} & \cdots & \frac{\partial f_n}{\partial y_n} \end{pmatrix}_{n\times n}$, $\frac{\partial^2 \mathbf{f}(\mathbf{y})}{\partial \mathbf{y}^2} = \begin{pmatrix} \frac{\partial^2 f_1}{\partial \mathbf{y}\partial y_1} & \frac{\partial^2 f_1}{\partial \mathbf{y}\partial y_2} & \cdots & \frac{\partial^2 f_1}{\partial \mathbf{y}\partial y_n} \\ \frac{\partial^2 f_2}{\partial \mathbf{y}\partial y_1} & \frac{\partial^2 f_2}{\partial \mathbf{y}\partial y_2} & \cdots & \frac{\partial^2 f_2}{\partial \mathbf{y}\partial y_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f_n}{\partial \mathbf{y}\partial y_1} & \frac{\partial^2 f_n}{\partial \mathbf{y}\partial y_2} & \cdots & \frac{\partial^2 f_n}{\partial \mathbf{y}\partial y_n} \end{pmatrix}_{n\times n^2}$, $\mathrm{diag}(\mathbf{y}-\hat{\mathbf{y}}, ..., \mathbf{y}-\hat{\mathbf{y}}) = \mathbb{Y}(t) - \hat{\mathbb{Y}}(t|t_{k-1})$.

We note that $\frac{\partial^2 \mathbf{f}(\mathbf{y})}{\partial \mathbf{y}^2}$ and $\mathbb{Y}(t)-\hat{\mathbb{Y}}(t|t_{k-1})$ are $n\times n$ block matrices whose entries are n-dimensional row vectors and column vectors, respectively. Moreover, $\frac{\partial^2 \mathbf{f}(\mathbf{y})}{\partial \mathbf{y}^2}$ is referred to as vector-valued Hessian matrix, and $\mathbb{Y}(t) - \hat{\mathbb{Y}}(t|t_{k-1})$ is a diagonal matrix defined in (5.13).

Following these definitions and notations, we define the a-posterior state estimate $\hat{x}(t_k|t_k)$ and a-posterior covariance estimate $P(t_k|t_k)$ as a quadratic approximation of the form

$$\begin{aligned} \hat{x}(t_k|t_k) &= A_0 + A_1(y(t_k) - \hat{y}(t_k|t_{k-1})) + A_2(\mathbb{Y}(t) - \hat{\mathbb{Y}}(t|t_{k-1}))(y(t_k) - \hat{y}(t_k|t_{k-1})) \\ P(t_k|t_k) &= B_0 + B_1(y(t_k) - \hat{y}(t_k|t_{k-1}))(y(t_k) - \hat{y}(t_k|t_{k-1}))^T, \end{aligned} \tag{5.17}$$

where A_0 is an $n \times 1$ matrix (column vector), A_1 is an $n \times n$ matrix, A_2 is an $n \times n$ block matrix whose entries are $1 \times n$ matrix (row vector), B_0 and B_1 are square $n \times n$ matrices.

In order to develop an algorithm for $\hat{x}(t_k|t_k)$ and $P(t_k|t_k)$, we need to solve for the A_i and B_i for $i = 0, 1$, A_2. For this purpose, we need to evaluate each quantity in (5.11)-(5.12). We use the multi-dimensional extension of Stirling's interpolation formula discussed in Magnus [78] and Luo [75] to approximate the state drift, diffusion and the observation functions in (5.9) up to the second order.

Using the second-order polynomials, we define the multidimensional interpolation formula as

$$\begin{aligned} \mathbf{f}(x) &= \mathbf{f}(\hat{x}) + \tilde{D}_{\Delta x}\mathbf{f}(\hat{x}) + \tfrac{1}{2}\tilde{D}^2_{\Delta x}\mathbf{f}(\hat{x}), \\ \mathbf{g}(x) &= \mathbf{g}(\hat{x}) + \tilde{D}_{\Delta x}\mathbf{g}(\hat{x}) + \tfrac{1}{2}\tilde{D}^2_{\Delta x}\mathbf{g}(\hat{x}), \\ \mathbf{h}(x) &= \mathbf{h}(\hat{x}) + \tilde{D}_{\Delta x}\mathbf{h}(\hat{x}) + \tfrac{1}{2}\tilde{D}^2_{\Delta x}\mathbf{h}(\hat{x}), \end{aligned} \tag{5.18}$$

where the operator $\tilde{D}_{\Delta x}$, and $\tilde{D}^2_{\Delta x}$ are described in [78] and are defined by

$$\begin{aligned} \tilde{D}_{\Delta x} &= \tfrac{1}{h}\left(\sum_{p=1}^{n} \Delta x_p \mu_p \delta_p\right), \\ \tilde{D}^2_{\Delta x} &= \tfrac{1}{h^2}\left(\sum_{p=1}^{n} \Delta x_p^2 \delta_p^2 + \sum_{p=1}^{n}\sum_{\substack{q=1\\ q\neq p}}^{n} \Delta x_p \Delta x_q (\mu_p\delta_p)(\mu_q\delta_q)\right), \end{aligned} \tag{5.19}$$

where Δx, δ_p and μ_p are defined by

$$\begin{array}{lcl} \Delta x & = & x - \hat{x}, \\ \delta_p \mathbf{f}(\hat{x}) & = & \mathbf{f}\left(\hat{x} + \frac{h}{2} e_p\right) - \mathbf{f}\left(\hat{x} - \frac{h}{2} e_p\right), \\ \mu_p \mathbf{f}(\hat{x}) & = & \frac{1}{2}\left[\mathbf{f}\left(\hat{x} + \frac{h}{2} e_p\right) + \mathbf{f}\left(\hat{x} - \frac{h}{2} e_p\right)\right], \end{array} \tag{5.20}$$

and $h > 0$ is the step size, e_p is the pth unit vector.

Using the Cholesky transformation, we transform x to a variable z which is mutually uncorrelated. Following [78], we write

$$\begin{array}{lcl} z & = & S_x^{-1} x, \\ \tilde{\mathbf{f}}(z) & = & \mathbf{f}(S_x z) = \mathbf{f}(x). \end{array} \tag{5.21}$$

From (5.21), (5.18) reduces to

$$\begin{array}{lcl} \tilde{\mathbf{f}}(z) & = & \tilde{\mathbf{f}}(\hat{z}) + \tilde{D}_{\Delta z} \tilde{\mathbf{f}}(\hat{z}) + \frac{1}{2} \tilde{D}^2_{\Delta z} \tilde{\mathbf{f}}(\hat{z}), \\ \tilde{\mathbf{g}}(z) & = & \tilde{\mathbf{g}}(\hat{z}) + \tilde{D}_{\Delta z} \tilde{\mathbf{g}}(\hat{z}) + \frac{1}{2} \tilde{D}^2_{\Delta z} \tilde{\mathbf{g}}(\hat{z}), \\ \tilde{\mathbf{h}}(z) & = & \tilde{\mathbf{h}}(\hat{z}) + \tilde{D}_{\Delta z} \tilde{\mathbf{h}}(\hat{z}) + \frac{1}{2} \tilde{D}^2_{\Delta z} \tilde{\mathbf{h}}(\hat{z}). \end{array} \tag{5.22}$$

Let σ_i represent the ith moment of an arbitrary element in Δz. We shall use the interpolation approximations (5.22) to evaluate the expressions in (5.11)-(5.12). For this purpose, we prove the following Lemma.

Assumption $\mathcal{B}$:

As discussed in Magnus [78], we assume Δz to be iid Gaussian. Hence,

$$\sigma_{2i-1} = 0, \quad i \in \mathbb{N}. \tag{5.23}$$

LEMMA 5.1 *Under the Assumption $\mathcal{B}$, we have*

$$\begin{array}{lcl} r_{0,2}(t_k|t_{k-1}) & = & \dfrac{\sigma_2}{h^2} \displaystyle\sum_{p=1}^{n} \mu_p \delta_p \tilde{\boldsymbol{h}}(\hat{z}) \mu_p \delta_p \tilde{\boldsymbol{h}}(\hat{z})^T + \dfrac{\sigma_4 - \sigma_2^2}{4h^4} \displaystyle\sum_{p=1}^{n} \delta_p^2 \tilde{\boldsymbol{h}}(\hat{z}) \delta_p^2 \tilde{\boldsymbol{h}}(\hat{z})^T \\ & & + \dfrac{\sigma_2^2}{4h^4} \displaystyle\sum_{\substack{p,q=1 \\ q \neq p}}^{n} \mu_p \delta_p \mu_q \delta_q \tilde{\boldsymbol{h}} \mu_p \delta_p \mu_q \delta_q \tilde{\boldsymbol{h}}^T + R \\ r_{1,1}(t_k|t_{k-1}) & = & \dfrac{\sigma_2}{h} \displaystyle\sum_{p=1}^{n} S_x \left(\mu_p \delta_p \tilde{\boldsymbol{h}}(\hat{z})\right)^T \\ r_{1,2}(t_k|t_{k-1}) & = & S_x \left(D(t_k|t_{k-1})_{\{\tilde{\boldsymbol{h}}_i, \tilde{\boldsymbol{h}}^T\}}\right)_{1 \le i \le n} + \hat{x}(t_k|t_{k-1}) r_{0,2}(t_k|t_{k-1})_{\{\tilde{\boldsymbol{h}}_i, \tilde{\boldsymbol{h}}^T\}} \end{array}$$

$$
\begin{aligned}
r_{1,3}(t_k|t_{k-1}) &= S_x J - 2S_x E(t_k|t_{k-1}) - 2r_{1,1}C(t_k|t_{k-1})C^T - r_{1,1}C^T C(t_k|t_{k-1}) \\
&\quad -S_x D(t_k|t_{k-1})_{\{\tilde{\boldsymbol{h}}^T,\tilde{\boldsymbol{h}}\}} C^T, \\
r_{2,2}(t_k|t_{k-1}) &= S_x \left(Q_{i,j}\right)_{\substack{1\le i\le n\\ 1\le j\le n}} \\
r_{0,3}(t_k|t_{k-1}) &= \left(L_{i,j}\right)_{\substack{1\le i\le n\\ 1\le j\le n}} \\
r_{0,4}(t_k|t_{k-1}) &= \mathbb{E}\left[\mathcal{A}\mathcal{A}^T\mathcal{A}\mathcal{A}^T|Y_{t_{k-1}}\right] - \mathbb{E}\left[\mathcal{A}\mathcal{A}^T\mathcal{A}C^T|Y_{t_{k-1}}\right] - \mathbb{E}\left[\mathcal{A}\mathcal{A}^T C\mathcal{A}^T|Y_{t_{k-1}}\right] \\
&\quad +\mathbb{E}\left[\mathcal{A}\mathcal{A}^T CC^T|Y_{t_{k-1}}\right] - \mathbb{E}\left[\mathcal{A}C^T\mathcal{A}\mathcal{A}^T|Y_{t_{k-1}}\right] + \mathbb{E}\left[\mathcal{A}C^T\mathcal{A}C^T|Y_{t_{k-1}}\right] \\
&\quad +\mathbb{E}\left[\mathcal{A}C^T C\mathcal{A}^T\right] - \mathbb{E}\left[\mathcal{A}C^T CC^T|Y_{t_{k-1}}\right] \\
\sigma_{Y2}(t_k|t_{k-1}) &= \left(F_i\right)_{1\le i\le n},
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{A} &= \tilde{D}_{\Delta z}\tilde{\boldsymbol{h}}(\hat{z}) + \frac{1}{2}\tilde{D}^2_{\Delta z}\tilde{\boldsymbol{h}}(\hat{z}) + v \\
C &= \frac{\sigma_2}{2h^2}\sum_{p=1}^{n}\delta_p^2\tilde{\boldsymbol{h}}(\hat{z}), \\
F_i &= \frac{\sigma_2}{h^2}\sum_{p=1}^{n}\left(\mu_p\delta_p\tilde{\boldsymbol{h}}(\hat{z})\right)\mu_p\delta_p\tilde{\boldsymbol{h}}_i(\hat{z}) + \frac{\sigma_4-\sigma_2^2}{4h^4}\sum_{p=1}^{n}\left(\delta_p^2\tilde{\boldsymbol{h}}(\hat{z})\right)\delta_p^2\tilde{\boldsymbol{h}}_i(\hat{z}) \\
&\quad + \frac{\sigma_2^2}{4h^4}\sum_{p=1}^{n}\sum_{\substack{q=1\\ q\ne p}}^{n}\left(\mu_p\delta_p\mu_q\delta_q\tilde{\boldsymbol{h}}(\hat{z})\right)\mu_p\delta_p\mu_q\delta_q\tilde{\boldsymbol{h}}_i(\hat{z}) + e_i R_{i,i}
\end{aligned}
$$

$$
r_{0,2}(t_k|t_{k-1})_{\{\tilde{\boldsymbol{h}}_i,\tilde{\boldsymbol{h}}^T\}} = \left(\frac{\sigma_2}{h^2}\sum_{p=1}^{n}\mu_p\delta_p\tilde{\boldsymbol{h}}_i(\hat{z})\mu_p\delta_p\tilde{\boldsymbol{h}}^T(\hat{z}) + \frac{\sigma_4-\sigma_2^2}{4h^4}\sum_{p=1}^{n}\delta_p^2\tilde{\boldsymbol{h}}_i(\hat{z})\delta_p^2\tilde{\boldsymbol{h}}^T(\hat{z}) + \frac{\sigma_2^2}{4h^4}\sum_{\substack{p,q=1\\ q\ne p}}^{n}\mu_p\delta_p\mu_q\delta_q\tilde{\boldsymbol{h}}_i\mu_p\delta_p\mu_q\delta_q\tilde{\boldsymbol{h}}^T + R_{i,i}e^T\right)_{1\le i\le n},
$$

$\tilde{\boldsymbol{h}} \equiv \tilde{\boldsymbol{h}}(\hat{z}) = \tilde{\boldsymbol{h}}(\hat{z}(t_k|t_{k-1}))$, $r_{0,2}(t_k|t_{k-1}) = r_{0,2}(t_k|t_{k-1})_{\{\tilde{\boldsymbol{h}},\tilde{\boldsymbol{h}}^T\}}$, *and detailed expressions for* $r_{0,2}(t_k|t_{k-1})_{\{\tilde{\boldsymbol{h}},\tilde{\boldsymbol{h}}^T\}}$, $J(t_k|t_{k-1})$, $E(t_k|t_{k-1})$, $D(t_k|t_{k-1})_{\{\tilde{\boldsymbol{h}},\tilde{\boldsymbol{h}}^T\}}$, $D(t_k|t_{k-1})_{\{\tilde{\boldsymbol{h}}^T,\tilde{\boldsymbol{h}}\}}$, $D(t_k|t_{k-1})_{\{\tilde{\boldsymbol{h}}_i,\tilde{\boldsymbol{h}}^T\}}$, $Q_{i,j}$, $\mathbb{E}\left[\mathcal{A}\mathcal{A}^T\mathcal{A}\mathcal{A}^T|Y_{t_{k-1}}\right]$, $\mathbb{E}\left[\mathcal{A}\mathcal{A}^T\mathcal{A}C^T|Y_{t_{k-1}}\right]$, $\mathbb{E}\left[\mathcal{A}\mathcal{A}^T C\mathcal{A}^T|Y_{t_{k-1}}\right]$, $\mathbb{E}\left[\mathcal{A}\mathcal{A}^T CC^T|Y_{t_{k-1}}\right]$, $\mathbb{E}\left[\mathcal{A}C^T\mathcal{A}\mathcal{A}^T|Y_{t_{k-1}}\right]$, $\mathbb{E}\left[\mathcal{A}C^T\mathcal{A}C^T|Y_{t_{k-1}}\right]$, $\mathbb{E}\left[\mathcal{A}C^T C\mathcal{A}^T|Y_{t_{k-1}}\right]$, $\mathbb{E}\left[\mathcal{A}C^T CC^T|Y_{t_{k-1}}\right]$, *and* $L_{i,j}$ *are given in Appendix* $B.2$.

Proof. The proof is given in Appendix $B.3$. □

We can now use these values to solve for A_i, B_i, $i = 0, 1$, and A_2. The first step in the algorithm is to solve for A_i, B_i, $i = 0, 1$ and A_2 in (5.17). For this purpose, we use the following Lemma by following the description of the moment propagation procedure across the observations described in Jazwinski [53].

LEMMA 5.2 *Under the assumptions in Lemma* 5.1*, we have*

$$\begin{aligned} A_0(t_k|t_{k-1}) &= r_{1,0}(t_k|t_{k-1}) - A_2(t_k|t_{k-1})\sigma_{Y2}(t_k|t_{k-1}), \\ A_1(t_k|t_{k-1}) &= \left[r_{1,1}(t_k|t_{k-1}) - A_2(t_k|t_{k-1})r_{0,3}(t_k|t_{k-1})^T\right] r_{0,2}(t_k|t_{k-1})^{-1}, \\ A_2(t_k|t_{k-1}) &= T_1(t_k|t_{k-1})T_2^{-1}(t_k|t_{k-1}), \end{aligned} \tag{5.24}$$

where

$$\begin{aligned} T_1(t_k|t_{k-1}) &= r_{1,2}(t_k|t_{k-1}) - r_{1,0}(t_k|t_{k-1})\sigma_{Y2}^T(t_k|t_{k-1}) \\ &\quad - r_{1,1}(t_k|t_{k-1})r_{0,2}^{-1}(t_k|t_{k-1})r_{0,3}(t_k|t_{k-1}) \\ T_2(t_k|t_{k-1}) &= M_{0,2}(t_k|t_{k-1}) - \sigma_{Y2}(t_k|t_{k-1})\sigma_{Y2}^T(t_k|t_{k-1}) \\ &\quad - r_{0,3}^T(t_k|t_{k-1})r_{0,2}^{-1}(t_k|t_{k-1})r_{0,3}(t_k|t_{k-1}). \end{aligned}$$

Proof. Proof is in Appendix $B.4$. □

REMARK 7 If $A_2 = 0$, (5.2) reduces to

$$\begin{aligned} A_0 &= \hat{x}(t_k|t_{k-1}) \\ A_1 &= r_{1,1}(t_k|t_{k-1})r_{0,2}(t_k|t_{k-1})^{-1}. \end{aligned}$$

Now, we present a Lemma for finding B_0 and B_1.

LEMMA 5.3 *Under the assumptions in Lemma* 5.1*, we have*

$$\begin{aligned} B_1 &= \Big(N_2(t_k|t_{k-1})r_{0,2}^{-1}(t_k|t_{k-1}) - N_1(t_k|t_{k-1})\Big)\Big[r_{0,4}(t_k|t_{k-1})r_{0,2}^{-1}(t_k|t_{k-1}) \\ &\quad - r_{0,2}(t_k|t_{k-1})\Big]^{-1} \\ B_0 &= N_1(t_k|t_{k-1}) - B_1(t_k|t_{k-1})r_{0,2}(t_k|t_{k-1}), \end{aligned} \tag{5.25}$$

where

$$
\begin{aligned}
N_1 &= \mathbb{E}\left[(x(t_k)-\hat{x}(t_k|t_k))(x(t_k)-\hat{x}(t_k|t_k))^T|Y_{t_{k-1}}\right] \\
&= P(t_k|t_{k-1}) - r_{1,1}(t_k|t_{k-1})A_1^T - r_{1,2}(t_k|t_{k-1})A_2^T - A_1 r_{1,1}(t_k|t_{k-1})^T \\
&\quad -A_2 r_{1,2}(t_k|t_{k-1}))^T + (\hat{x}(t_k|t_{k-1})-A_0)(\hat{x}(t_k|t_{k-1})-A_0)^T \\
&\quad -(\hat{x}(t_k|t_{k-1})-A_0) r_{0,2}(t_k|t_{k-1})_{\{\tilde{\boldsymbol{h}}_i,\tilde{\boldsymbol{h}}^T\}} A_2 \\
&\quad +A_1 r_{0,2}(t_k|t_{k-1})A_1^T + A_1 r_{0,2}(t_k|t_{k-1})_{\{\tilde{\boldsymbol{h}}_i,\tilde{\boldsymbol{h}}^T\}} A_2^T \\
&\quad -A_2 r_{0,2}(t_k|t_{k-1})_{\{\tilde{\boldsymbol{h}}_i,\tilde{\boldsymbol{h}}^T\}}(\hat{x}(t_k|t_{k-1})-A_0) \\
&\quad +A_2 r_{0,3}(t_k|t_{k-1})A_1 + A_2 M_{0,2}(t_k|t_{k-1})A_2^T \\
N_2 &= \mathbb{E}\left[(x(t_k)-\hat{x}(t_k|t_k))(x(t_k)-\hat{x}(t_k|t_k))^T(y(t_k)-\hat{y}(t_k|t_k))\times\right. \\
&\quad \left.(y(t_k)-\hat{y}(t_k|t_k))^T|Y_{t_{k-1}}\right] \\
&= \mathbb{E}\left(\left[(x(t_k)-A_0)(x(t_k)-A_0)^T - (x(t_k)-A_0)(y(t_k)-\hat{y}(t_k|t_{k-1}))^T A_1^T\right.\right. \\
&\quad -(x(t_k)-A_0)(y(t_k)-\hat{y}(t_k|t_{k-1}))^T(\mathbb{Y}(t_k)-\hat{\mathbb{Y}}(t_k|t_{k-1}))^T A_2 \\
&\quad -A_1(y(t_k)-\hat{y}(t_k|t_{k-1}))(x(t_k)-A_0)^T \\
&\quad \left.+A_1(y(t_k)-\hat{y}(t_k|t_{k-1}))(y(t_k)-\hat{y}(t_k|t_{k-1}))^T A_1^T\right]\times \\
&\quad \left.(y(t_k)-\hat{y}(t_k|t_{k-1}))(y(t_k)-\hat{y}(t_k|t_{k-1}))^T|Y_{t_{k-1}}\right) \\
&= r_{2,2} + (\hat{x}(t_k|t_{k-1})-A_0)(\hat{x}(t_k|t_{k-1})-A_0)^T r_{0,2} \\
&- \mathbb{E}\left[(x(t_k)-\hat{x}(t_k|t_{k-1}))(y(t_k)-\hat{y}(t_k|t_{k-1}))^T A_1^T(y(t_k)-\hat{y}(t_k|t_{k-1}))\times\right. \\
&\quad \left.(y(t_k)-\hat{y}(t_k|t_{k-1}))^T|Y_{k-1}\right] \\
&- \mathbb{E}\left[(x(t_k|t_{k-1})-A_0)(y(t_k)-\hat{y}(t_k|t_{k-1}))^T A_1^T(y(t_k)-\hat{y}(t_k|t_{k-1}))\times\right. \\
&\quad \left.(y(t_k)-\hat{y}(t_k|t_{k-1}))^T|Y_{t_{k-1}}\right] \\
&- \mathbb{E}\left[A_1(y(t_k)-\hat{y}(t_k|t_{k-1}))(x(t_k)-\hat{x}(t_k|t_{k-1}))^T(y(t_k)-\hat{y}(t_k|t_{k-1}))\times\right. \\
&\quad \left.(y(t_k)-\hat{y}(t_k|t_{k-1}))^T|Y_{t_{k-1}}\right] \\
&- \mathbb{E}\left[A_1(y(t_k)-\hat{y}(t_k|t_{k-1}))(x(t_k|t_{k-1})-A_0)^T(y(t_k)-\hat{y}(t_k|t_{k-1}))\times\right. \\
&\quad \left.(y(t_k)-\hat{y}(t_k|t_{k-1}))^T|Y_{t_{k-1}}\right] \\
&+ \mathbb{E}\left[A_1(y(t_k)-\hat{y}(t_k|t_{k-1}))(y(t_k)-\hat{y}(t_k|t_{k-1}))^T A_1^T(y(t_k)-\hat{y}(t_k|t_{k-1}))\times\right. \\
&\quad \left.(y(t_k)-\hat{y}(t_k|t_{k-1}))^T|Y_{t_{k-1}}\right],\ \textit{and } \hat{x}(t_k|t_k) \textit{ is given in } (5.17).
\end{aligned}
$$

Proof. The proof is shown in Appendix $B.5$. $\square$

REMARK 8 If $B_1 = 0$,and $A_2 = 0$, then from (5.25) and Remark 7, we have

$$\begin{aligned} A_0 &= r_{1,0}(t_k|t_{k-1}) = \hat{x}(t_k|t_{k-1}), \\ A_1 &= r_{1,1}(t_k|t_{k-1}) r_{0,2}(t_k|t_{k-1})^{-1} \\ B_0 &= N_1 = P_{t_k|t_{k-1}} - A_1 r_{0,2}(t_k|t_{k-1}) A_1^T. \end{aligned}$$

Thus, the presented state and covariance algorithm includes the EKF scheme [82] as a special case.

5.3.1 Posterior Prediction of State and Covariance of Nonlinear System

A final step in the recursive algorithm is to predict the state $\hat{x}(t_{j+1}|t_j)$ and state variance $P(t_{j+1}|t_j)$ at the time of the following measurement. Using (5.9), the definition of $P(t|t_{k-1})$ in (5.11), and (5.18),we have

$$\begin{aligned} \hat{x}(t_{k+1}|t_k) &= \hat{x}(t_k|t_k) + \left[\tilde{\mathbf{f}}(\hat{z}(t_k|t_k)) + \tfrac{\sigma_2}{2h^2}\sum_{p=1}^{n}\delta_p^2\tilde{\mathbf{f}}(\hat{z}(t_k|t_k))\right]\Delta t, \\ P(t_{k+1}|t_k) &= P(t_k|t_k) + [\tfrac{\sigma_2}{h}\sum_{p=1}^{n}\mu_p\delta_p\tilde{\mathbf{f}}(\hat{z}(t_k|t_k))e_p^T S_x^T + \tfrac{\sigma_2}{h}\sum_{p=1}^{n} S_x e_p \mu_p\delta_p\tilde{\mathbf{f}}^T(\hat{z}(t_k|t_k)) \\ & \quad + \tfrac{\sigma_2}{h^2}\sum_{p=1}^{n}\mu_p\delta_p\tilde{\mathbf{f}}(\hat{z}(t_k|t_k))\mu_p\delta_p\tilde{\mathbf{f}}^T(\hat{z}(t_k|t_k)) \\ &+ \tfrac{\sigma_4-\sigma_2^2}{4h^2}\sum_{p=1}^{n}\delta_p^2\tilde{\mathbf{f}}(\hat{z}(t_k|t_k))\delta_p^2\tilde{\mathbf{f}}^T(\hat{z}(t_k|t_k)) \\ & \quad + \tfrac{\sigma_2}{2h^2}\tilde{\mathbf{g}}(\hat{z}(t_k|t_k))\sum_{p=1}^{n}\delta_p^2\tilde{\mathbf{g}}^T(\hat{z}(t_k|t_k)) \\ &+ \tfrac{\sigma_2^2}{4h^2}\sum_{\substack{p,q=1\\p\neq q}}^{n}\Big[\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{f}}(\hat{z}(t_k|t_k))\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{f}}^T(\hat{z}(t_k|t_k)) \\ & \quad + \mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{f}}(\hat{z}(t_k|t_k))\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{f}}^T(\hat{z}(t_k|t_k))\Big] \\ &+ \tfrac{\sigma_2}{h^2}\sum_{p=1}^{n}\mu_p\delta_p\tilde{\mathbf{g}}(\hat{z}(t_k|t_k))\mu_p\delta_p\tilde{\mathbf{g}}^T(\hat{z}(t_k|t_k)) + \tilde{\mathbf{g}}(\hat{z}(t_k|t_k))\tilde{\mathbf{g}}^T(\hat{z}(t_k|t_k))... \\ &+ \tfrac{\sigma_2}{2h^2}\sum_{p=1}^{n}\delta_p^2\tilde{\mathbf{g}}(\hat{z}(t_k|t_k))\tilde{\mathbf{g}}^T(\hat{z}(t_k|t_k)) \\ & \quad + \tfrac{\sigma_4}{4h^2}\sum_{p=1}^{n}\delta_p^2\tilde{\mathbf{g}}(\hat{z}(t_k|t_k))\delta_p^2\tilde{\mathbf{g}}^T(\hat{z}(t_k|t_k)) \\ &+ \tfrac{\sigma_2^2}{4h^2}\sum_{\substack{p,q=1\\p\neq q}}^{n}\Big[\delta_p^2\tilde{\mathbf{g}}(\hat{z}(t_k|t_k))\delta_q^2\tilde{\mathbf{g}}^T(\hat{z}(t_k|t_k)) \\ & \quad + \mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{g}}(\hat{z}(t_k|t_k))\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{g}}^T(\hat{z}(t_k|t_k))\Big]\Delta t \\ &+ \tfrac{\sigma_2^2}{4h^2}\sum_{\substack{p,q=1\\p\neq q}}^{n}\Big[\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{g}}(\hat{z}(t_k|t_k))\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{g}}^T(\hat{z}(t_k|t_k))\Big]]\Delta t \end{aligned} \tag{5.26}$$

The one step prediction error

$$\Delta y(k) = y_k - \hat{y}(t_k|t_{k-1}), \tag{5.27}$$

is assumed to be normal with mean 0 and variance $r_{0,2}$. Hence, for N independent random observations, the Maximum Likelihood approach is equivalent to maximizing

$$L(\boldsymbol{\Theta}) = -\frac{1}{2}\sum_{k=1}^{N}\left[\frac{1}{2}\Delta y^T(k) r_{0,2}^{-1}(t_k|t_{k-1})\Delta y(k) + \log|r_{0,2}(t_k|t_{k-1})|\right], \tag{5.28}$$

where $\boldsymbol{\Theta}$ is the parameter space.

REMARK 9 The presented predicted algorithm extends the algorithm generated by the EKF approach in a systematic way. We further remark that the second order estimation for nonlinear stochastic systems can be extended to higher order estimation. The scheme is highly complex mathematical expressions. Further detailed examination (applicability/computational, feasibility, et.c) is under investigation.

5.3.2 Algorithm

We describe the algorithm used in the computation of the estimates for nonlinear log-spot price stochastic differential equation (5.29) in Appendix $B.1$.

5.4 Some Results: Natural Gas

In this section, we give the parameter estimates for the stochastic differential equation (5.9). We consider the nonlinear stochastic differential equation that was developed for describing continuous time stochastic dynamic model of energy commodities log-spot price processes in (4.11),

$$\begin{aligned} dx_1 &= \mu(x_1+\kappa_0)(\kappa_2-x_1)dt + \delta(\kappa_2-x_1)dW_1(t), \;\; x_1(t_0)=x_{10}, \\ dx_2 &= \gamma(x_2+\kappa_1)(x_1-x_2)dt + \sigma(t,x_{2t})(x_2+\kappa_1)dW_2(t), \;\; x_2(t_0)=x_{02}. \\ y(t) &= x(t)+v(t). \end{aligned} \tag{5.29}$$

It follows from (5.29) that

$$\mathbf{f}(x;\boldsymbol{\theta}) = \begin{pmatrix} \mu(x_1+\kappa_0)(\kappa_2-x_1) \\ \gamma(x_2+\kappa_1)(x_1-x_2) \end{pmatrix}, \mathbf{g}(x;\boldsymbol{\theta}) = \begin{pmatrix} \delta(\kappa_2-x_1) & 0 \\ 0 & \sigma(t,x_{2t})(x_2+\kappa_1) \end{pmatrix},$$

and $x = \{x_1, x_2\}^T$, where $\mu > 0$, $\gamma > 0$, $\kappa_0 \geq 0$, $\kappa_1 \geq 0$, $\kappa_2 > 0$, $\delta > 0$, $\sigma > 0$, v is a white noise, and $\mathbf{W} = \{W_1, W_2\}^T$, W_1 and W_2 are independent Wiener processes. This model governs the price for energy commodity at time t. $x_2(t)$ is the nonseasonal log of spot price at a time t and

$x_1(t)$ describes a mean process of non-seasonal log spot price at time t. The model (5.29) follows the principle of demand and supply processes which suggest that the price of a energy commodity will remain within a given finite lower and upper bounds $\kappa_1 > 0$ and $\kappa_2 > 0$, respectively. In this case, κ_2 characterizes the fixed cost, $(x_1(t) + \kappa_0)(\kappa_2 - x_1)$ characterizes the market potential for $x_1(t)$ per unit of time at a time t. We note that the first component of (5.29) has a unique non-zero equilibrium κ_2. Moreover, we observe that whenever the price x_1 lies above κ_2, there is a tendency for the price to fall and whenever the price is below κ_2, the price rises back. Hence, κ_2 is the equilibrium of the first component of (5.29). Furthermore, μ and γ are the rate of mean reversion for x_1 and x_2 respectively, δ and σ are the volatility for x_1 and x_2 respectively.

We apply this model to the Henry-Hub natural gas data set [24]. We use the Henry-Hub natural gas spot price data set [24] for the observation data for x_2. We generate observation data for x_1 from the forward price $F(t, T)$ at time t of an energy goods with maturity at time T. We define the forward price as

$$F(t, T) = \mathbb{E}_{\mathcal{P}}\left(x_2(T)\right). \tag{5.30}$$

By definition, $x_1(t)$ is the expected log-spot price, which in this case is the observation data $F(t, T)$. We use Henry-Hub natural gas observed future price at a time t with delivery time T.

The existence and uniqueness of the solution of (5.29) is given in Chapter 4.

The initial state of the model is $\hat{x}_1(t_1|t_0) = 1.23$, $\hat{x}_2(t_1|t_0) = 1.456$, $P(t_1|t_0) = \begin{pmatrix} 0.1182 & 0 \\ 0 & 0.22 \end{pmatrix}$.

Table (4) shows the parameter estimates of Henry Hub daily natural gas.

Table 4: Estimated Parameters of (5.29) for Henry Hub daily natural gas spot prices (20 run average)

μ	γ	κ_0	κ_1	κ_2	δ	σ
1.6	1.78	.69	.56	1.5	0.65	0.47

Table 4 shows the estimates of the parameters of (5.29).

Furthermore, we show some of the estimates of the simulations for the modified extended Kalman filter (MEKF) scheme compared with the usual EKF scheme.

Table 5: Simulation estimates for Henry Hub data [24] using the MEKF and EKF scheme.

Data		Modified EKF	EKF	Modified EKF Error	EKF Error
t (days)	Real data	Simulated data	Simulated data	$Real - Simulated$	$Real - Simulated$
20	2.6990	2.6739	2.6651	0.0251	0.0339
21	2.7590	2.6649	2.7232	0.0941	0.0358
22	2.6590	2.6523	2.6975	0.0067	-0.0385
23	2.7420	2.7544	2.6265	-0.0124	0.1155
24	2.5620	2.5809	2.5338	-0.0189	0.0282
25	2.4950	2.4836	2.4779	0.0114	0.0171
26	2.540	2.5379	2.6115	0.0021	-0.0715
27	2.5920	2.5948	2.6037	-0.0028	-0.0117
28	2.5700	2.5955	2.5451	-0.0255	0.0249
29	2.5410	2.6038	2.5701	-0.0628	-0.0291
30	2.6180	2.5844	2.6920	0.0336	-0.0740
31	2.5640	2.6071	2.6563	-0.0431	-0.0923
32	2.6670	2.6819	2.6118	-0.0149	0.0552
33	2.6330	2.6251	2.6613	0.0079	-0.0283
34	2.5150	2.5311	2.6376	-0.0161	-0.1226
35	2.5300	2.5104	2.6580	0.0196	-0.1280
...	...	...	...	...	...
255	9.8190	7.9849	4.6175	1.8341	5.2015
256	9.1280	8.1730	4.6536	0.9550	4.4744
257	8.7080	8.4447	4.7375	0.2633	3.9705
258	8.4720	8.6903	4.6959	-0.2183	3.7761
259	8.1030	9.3793	4.8368	-1.2763	3.2662
260	6.9090	6.5429	4.7359	0.3661	2.1731
261	7.1360	6.8047	4.8381	0.3313	2.2979
262	7.4590	6.6625	4.5907	0.7965	2.8683
263	7.4570	6.6334	4.5749	0.8236	2.8821
264	6.9460	6.5458	4.5208	0.4002	2.4252
265	7.1150	5.8724	4.6052	1.2426	2.5098
266	7.2700	5.7080	5.0362	1.5620	2.2338
267	7.2560	5.4032	4.9095	1.8528	2.3465
268	6.2930	5.3971	4.7343	0.8959	1.5587
269	6.2930	5.5606	5.0757	0.7324	1.2173
270	6.2930	5.6742	5.4394	0.6188	0.8536
...	...	...	...	...	...
...	...	...	...	...	...

Data		Modified EKF	EKF	Modified EKF Error	EKF Error
t (days)	Real data	Simulated data	Simulated data	$Real - Simulated$	$Real - Simulated$
...	...	...	...	...	...
...	...	...	...	...	...
...	...	...	...	...	...
315	5.4220	5.5496	5.6280	-0.1276	-0.2060
316	5.3880	5.2974	5.3527	0.0906	0.0353
317	5.4770	5.5581	5.1236	-0.0811	0.3534
318	5.5590	5.6172	5.6023	-0.0582	-0.0433
319	5.3850	5.5434	4.8462	-0.1584	0.5388
320	5.3810	5.5147	4.6067	-0.1337	0.7743
321	5.5160	5.3233	6.1508	0.1927	-0.6348
322	5.2480	5.5923	4.6280	-0.3443	0.6200
323	5.1480	5.1522	4.4990	-0.0042	0.6490
324	5.1010	5.2694	4.4632	-0.1684	0.6378
325	5.1280	5.0945	4.3613	0.0335	0.7667
326	5.1250	4.9006	4.3357	0.2244	0.7893
327	5.0780	4.6212	4.4024	0.4568	0.6756
328	4.9810	4.4826	4.4873	0.4984	0.4937
329	4.8910	4.3396	4.4108	0.5514	0.4802
330	4.8670	4.5609	4.5258	0.3061	0.3412
...	...	...	...	...	...
...	...	...	...	...	...
...	...	...	...	...	...
1040	5.4430	5.5370	5.3943	-0.0940	0.0487
1041	5.3930	5.3264	5.0916	0.0666	0.3014
1042	5.4380	5.1881	5.7584	0.2499	-0.3204
1043	5.3970	5.3663	5.7657	0.0307	-0.3687
1044	5.6430	5.5917	6.0480	0.0513	-0.4050
1045	5.5960	5.6060	5.8662	-0.0100	-0.2702
1046	5.7180	5.3299	5.6337	0.3881	0.0843
1047	5.6880	5.2528	5.1967	0.4352	0.4913
1048	5.7220	5.2462	5.0721	0.4758	0.6499
1049	5.6310	5.3174	4.8923	0.3136	0.7387
1050	5.5820	5.3509	5.0484	0.2311	0.5336
1051	5.5460	5.6582	4.9531	-0.1122	0.5929
1052	5.5300	5.8134	6.1617	-0.2834	-0.6317
1053	5.4290	5.7149	5.8370	-0.2859	-0.4080
1054	5.3360	5.5240	5.6794	-0.1880	-0.3434
1055	5.3950	5.4978	4.7768	-0.1028	0.6182

Table 5 shows the real data sets, estimated simulation results for the Modified EKF scheme and the usual EKF scheme. The estimated error is calculated by subtracting the simulated estimates from the real data set.

We show the graph of the real and simulation results using MEKF scheme.

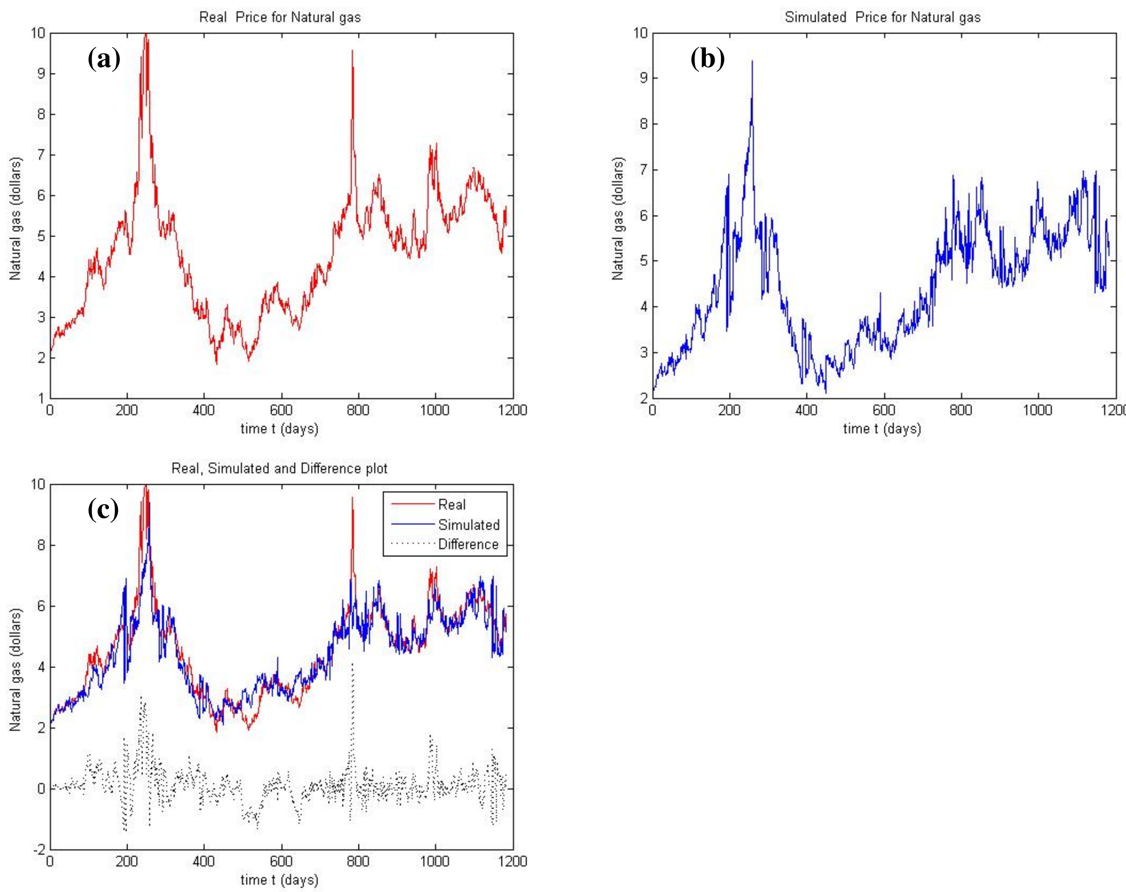
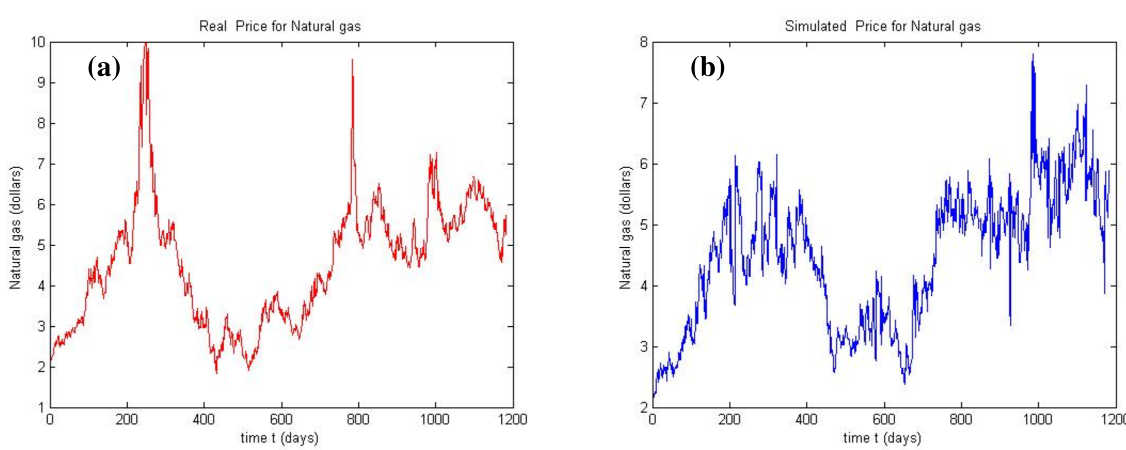


Figure 5.: Real and Simulated price for Natural gas data set [24] using Modified EKF scheme

Furthermore, we show the graph of the real and simulation results using EKF scheme.



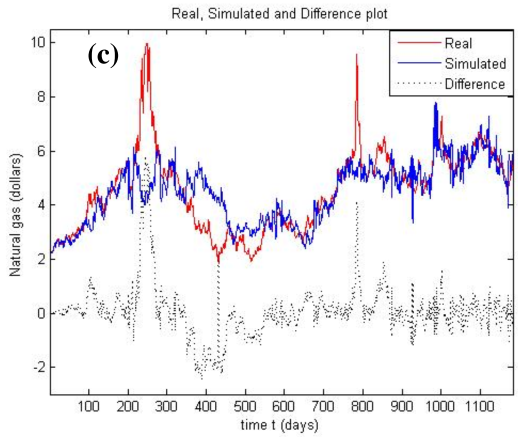
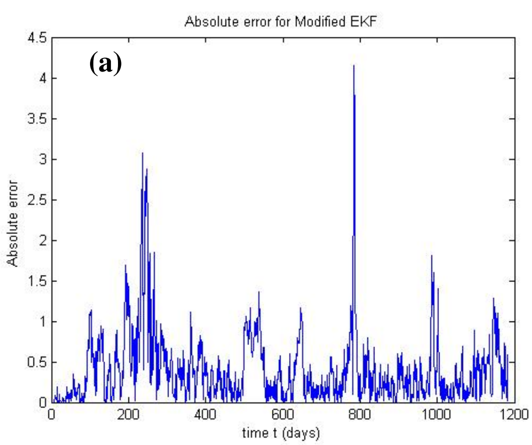


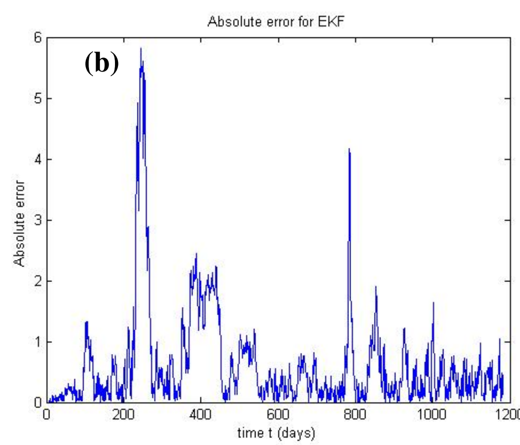
Figure 6.: Real and Simulated price for Natural gas data set [24] using EKF scheme

Figure 5 (a) shows the graph of the real natural gas data set, Figure 5 (b) shows the simulated price using Modified EKF scheme, and Figure 5 **(c)** shows the combination of the real, simulated and difference of the real and simulated price of the natural gas data set using the modified extended Kalman filter second order estimation scheme.

Figure 6**(a)** shows the graph of the real natural gas data set, Figure 6 (b) shows the simulated price the usual ordinary EKF scheme, and Figure 6 **(c)** shows the combination of the real, simulated and difference of the real and simulated price of the natural gas data set [24] using the usual ordinary EKF scheme.

The following graph show the absolute error of the simulation result using the MEKF and EKF scheme.





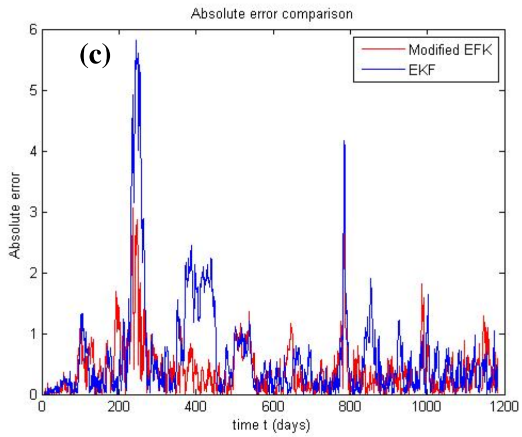


Figure 7.: Absolute error estimate using natural gas data set [24]

Figure 7 **(a)** shows the absolute error of the simulations of natural gas data set using the modified extended Kalman filter scheme, Figure 7 (b) shows the absolute error of the simulations of natural gas data set using the usual extended Kalman filter scheme, and Figure 7 **(c)** shows the comparison of the absolute error for the modified and usual EKF scheme.

REMARK 10 We further remark that all codes are written in Matlab. To compute the maximization $\arg\min L(\mathbf{\Theta})$ in the algorithm, we use the Nelder-Mead Simplex Method developed in Matlab. Maximizing (5.28) is equivalent to minimizing

$$L(\mathbf{\Theta}) = \frac{1}{2}\sum_{k=1}^{N}\left[\frac{1}{2}\Delta y^T(k) r_{0,2}^{-1}(t_k|t_{k-1})\Delta y(k) + \log|r_{0,2}(t_k|t_{k-1})|\right]. \tag{5.31}$$

REMARK 11 It is clear from Figures 5 and 6 that the presented scheme is superior than the EKF approach. This shows that the modified extended Kalman filter does in fact reduce the magnitude of error tremendously. Furthermore, the modified extended Kalman filter scheme was able to capture the upward price spike in the neighborhood of time $t = 250$ days better than the EKF scheme. Both scheme were not able to capture the upward spike around the time $t = 800$ days. This might be as a result of the kind of model we are using to describe the dynamics of the natural gas data set. The upward spike in price at these region was due to the decline in production of natural gas and the increase in demand for electricity generation. We will like to also mention one disadvantage with this scheme. It is computational intensive.

Chapter 6

Discrete Time Dynamic Model of Statistics Process and Applications

6.1 Introduction

Recently, several models have been developed to investigate the volatility process described by stochastic differential equations [140] and stochastic difference equations [38]. It is well-recognized that volatility is predictable in many asset markets [9]. Moreover, it is observed that the volatility predictability varies significantly. Engle [38] developed a class of discrete-time models where the variance depends on the past history of state of commodity/service. Bollerslev [9] generalized models in [38] to the GARCH(p,q).

Using the concept of moving average, the estimate for the variance of a general statistics from a stationary sequence is obtained [13]. Employing the batched mean, the grand mean of the individual batch mean and introducing ASAP3 [122], it is shown that ASAP3 fits AR(1) time series model to the batch mean, and it provides better technique for points and confidence-interval estimators.

It is well known and well recognized [33, 82, 118] that the Kalman filtering approach for the system parameter and state estimation problems is based on the continuous time coupled system of state dynamic and observation systems. Using the batched mean and the first order iterative process for $\bar{X}_n$ [137], a first order iterative process [137] is developed to estimate the population variance from a given time series data set.

For the past 40 years, researchers [7, 15, 21, 33, 44, 45, 47, 55, 86, 87, 89, 95, 103, 104, 105, 106, 107, 116, 118, 122] have paid lot of attention for estimating continuous-time dynamic models from discrete time data sets. The Generalized Method of Moments (GMM) developed by Hansen [44], and its extensions [21, 45, 47, 55] have played a significant role in the literature related to the parameter and state estimation problems in linear and nonlinear stochastic dynamic processes. Under the continuous-time dynamic and discrete time data collection processes, the GMM and its extensions/generalizations consist of : 1. Stochastic differential equations of It$\hat{o}$-Doob type, 2. Euler-type discretization scheme, 3. the general moment function, 4. minimizing functional or objective criterion function [44, 47].

The most of the existing parameter and state estimation techniques except the Kalman filtering are centered around the usage of either overall data sets [21, 45, 47, 55], or batched data sets [13], or local data set [107] drawn on an interval of finite length T. This leads to an overall parameter estimate on the interval of T. In this work, the presented approach is focused on the local moving lagged restriction of a finite sequence of a data set drawn at a partition P of finite interval of length T to a subpartition of P of moving subinterval $[t_{k-m_k}, t_{k-1}]$ of the interval. Moreover, using the lagged adaptive process, the present work initiates the technique to estimate the parameter and state at each data point for the given data set. Of course, these parameter estimates depend on the local admissible lagged finite restricted sequence of data. As the sub-partition moves from left to right, the approach provides a more lagged data subsets. In fact, the available lagged data subset at the previous time is a subset of the available lagged data subset at the subsequent times. The characteristics of this approach reduces the local error between a simulated value of the state of the system corresponding to the local available lagged restricted sequence of data under subpartition and pre determined performance criterion. We finally note that at the left end point of data simulation interval, without loss in generality, it is assumed that there is at least three data points that are assumed to be close enough to the true values of solution process of continuous dynamic process. In general, this is assured by the uniqueness and continuous dependence of solution process with respect to the initial data $(t_0, \boldsymbol{\varphi}_0)$ (for delay stochastic differential equation) and $(t_0, \mathbf{y}_0)$ (in the absence of delay stochastic differential equation) [70]. Moreover, as the location of data point approaches close to the right end point of the time interval, the local admissible lagged finite restricted sequence approaches to the given data set. We remark that this situation does not affect the computational ability. This is due to the fact that as the longativity of the past history approaches to the given data set, its influence diminishes. In fact, simulation value approaches to the saturation level under the performance criterion.

The presented local lagged adapted GMM method is based on the: 1. development of stochastic mathematical model of continuous time dynamic process [69, 70], 2. utilizing Euler-type discretized scheme [58] for the stochastic model in 1, 3. developing discrete time interconnected dynamic model for statistic process, 4. employing lagged adaptive expectation process [88] for developing generalized moment equations, 5. conceptual computational parameter estimation problem, 6. conceptual computational state simulation scheme, and 7. mean square ϵ-sub optimal procedure.

The present work is motivated by parameter and state estimation problems of continuous time nonlinear stochastic dynamic model of energy commodity markets described in (4.11). The purpose of the parameter and state estimation problems is for model validation rather than model mis-specification [21]. For the continuous-time dynamic model validation, we need to utilize the existing real world data set. Of course, the real world data set is drawn/recorded at discrete-time on a time interval of finite length. In view of this, employing the stochastic numerical approximation scheme [58], we approximate the continuous time stochastic differential equations. In almost real world dynamic modeling problems [64, 69, 70, 88], future states of continuous time dynamic processes are influenced by the states past history and response/reaction time delay processes to present states [64, 88]. Under this assumption and using the concept of lagged adaptive expectation process [47, 88], we formulate a discrete-time observation system. In fact, the discrete-time dynamic models depend on the past history of the state of a system [59]. By using the method of moments [14], and the constructed observation system, we estimate the state and its parameters. This idea leads to the development of interconnected disctete-time dynamic model of local sample mean and variance statistic processes. One of the by-products of the discrete-time sample variance statistic process is that it provides an alternative approach to the GARCH(1,1) model [9, 10]. Furthermore, the usage of the continuous-time stochastic dynamic model [69, 70], lagged expectation process, m_k- local lagged generalized method of moments, and interconnected discrete-time dynamic model of local sample mean and variance statistics processes lead to an alternative innovative method of state and parameter estimation problems for continuous-time dynamic models described by stochastic differential equations. The developed method is referred as local lagged adapted generalized method of moments (LLGMM). The numerical approximation process and simulation processes need to be synchronized with the existing data collection process. Using a schedule synchronization process, the concepts of local admissible sample/data observation size, local admissible finite conditional restriction sequence of data set are introduced. We estimate the parameters locally and then determine the local ϵ-sub-optimal simulated state estimates. In fact, our approach is more suitable and robust for forecasting problem. It also provides upper and lower bounds for the forecasted state of the system.

The organization of this study is as follows:

In Section 6.2, we derive a discrete time dynamic model for sample mean and variance processes. We introduce a new concept of parameter and state estimation techniques. This new concept is motivated by the parameter and state estimation problems of continuous time non-liner stochastic dynamic process. In Section 6.3, we construct observation system from a nonlinear stochastic functional differential equations. In addition, using the method of moments [14], in the context of lagged adaptive expectation process [88], we briefly outline a procedure to estimate the state parameters locally. The conceptual computational and simulation schemes are presented in Section 6.4. Moreover, a conceptual Matlab code and its implementation scheme are designed. The usefulness of computational algorithm is illustrated by applying the code to four energy commodity data sets, U.S. Treasury Bill Yield Interest Rate data set, and U. S. Eurocurrency Exchange Rate data set for the state and parameter estimation problems. Moreover, we compare the usage of GARCH(1,1) model with the presented model. Furthermore, we compare our simulated volatility U.S. Treasury Bill Yield Interest rate data with the simulated work of Chan et al [15].

6.2 Derivation of Discrete Time Dynamic Model for sample mean and variance Processes.

In this section, we use the idea of moving average to derive an algorithm for the mean and variance of sample sequences with respect to a continuous stochastic process. The development of idea and model of statistic for mean and variance processes is motivated by the state and parameter estimation problems of continuous time nonlinear stochastic dynamic model of the energy commodity market described in (4.11). In addition, the problem of price forecasting of energy goods is also addressed. For this purpose, we need to introduce a few definitions and notations.

Let τ and γ be finite constant time delays such that $0 < \gamma \leq \tau$. Here, τ characterizes the influence of the past performance history of state of dynamic process, and γ describes the reaction or response time delay. In general, these time delays are unknown and random variables. These types of delay play a role in developing mathematical models of continuous time [64] and discrete time [59, 88] dynamic processes. Based upon the practical nature of data collection process, it is essential to either transform these time delays into positive integers or design the data collection schedule in relations with these delays. For this purpose, we describe the discrete version of time delays of τ and γ as

$$r = \left[\left|\frac{\tau}{\Delta t_i}\right|\right] + 1, \text{ and } q = \left[\left|\frac{\gamma}{\Delta t_i}\right|\right] + 1, \tag{6.1}$$

respectively. Moreover, for the sake of simplicity, we assume that $0 < \gamma < 1$ (q=1).

DEFINITION 6.2.1 *Let x be a continuous time stochastic process defined on an interval $[-\tau, T]$ into $\mathbb{R}$, for some $T > 0$. For $t \in [-\tau, T]$, let $\mathcal{F}_t$ be an increasing sub-sigma algebra of a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ for which $x(t)$ is $\mathcal{F}_t$ measurable. Let P be a partition of $[-\tau, T]$ defined* *by*

$$P := \{t_i = t_0 + i\Delta t\}, \ \textit{for} \ \ i \in I(-r, N), \tag{6.2}$$

where $\Delta t = \frac{T-t_0}{N}$ *and* $I(a,b)$ *is defined by* $I(a,b) = \{j \in \mathbb{Z} \,|a \le j \le b\}$.

Let $\{x(t_i)\}_{i=-r}^{N}$ be a finite sequence corresponding to the stochastic process x and partition P in Definition 6.2.1. We further note that $x(t_i)$ is $\mathcal{F}_{t_i}$ measurable for $i \in I(-r, N)$. We recall the definition of forward time shift operator F [11] :

$$F^i x(t_k) = x(t_{k+i}). \tag{6.3}$$

In addition, let us denote $x(t_i)$ by x_i for $i \in I(-r, N)$.

DEFINITION 6.2.2 *For* $q = 1$ *and* $r \ge 1$, *each* $k \in I_0(N)$ *and each* $m_k \in I(2, r+k-1)$, *a partition* P_k *of closed interval* $[t_{k-m_k}, t_{k-1}]$ *is called local at time* t_k *and it is defined by*

$$P_k := t_{k-m_k} < t_{k-m_k+1} < ... < t_{k-1}. \tag{6.4}$$

Moreover, P_k *is referred as the* m_k-*point sub-partition of the partition* P *in (*6.2*) of the closed sub-interval* $[t_{k-m_k}, t_{k-1}]$ *of* $[-\tau, T]$.

DEFINITION 6.2.3 *For each* $k \in I_0(N)$ *and each* $m_k \in I(2, r+k-1)$, *a local finite sequence at a time* t_k *of the size* m_k *is restriction of* $\{x(t_i)\}_{i=-r}^{N}$ *to* P_k *in (*6.4*) [2], and it is defined by*

$$S_{m_k,k} := \{F^i x_{k-1}\}_{i=-m_k+1}^{0}. \tag{6.5}$$

As m_k *varies from* 2 *to* $k+r-1$, *the corresponding local sequence* $S_{m_k,k}$ *at* t_k *varies from* $\{x_i\}_{i=k-2}^{k-1}$ *to* $\{x_i\}_{i=-r+1}^{k-1}$. *As a result of this, the sequence defined in (*6.5*) is also called a* m_k*-local moving sequence. Furthermore, the average corresponding to the local sequence* $S_{m_k,k}$ *in (*6.5*) is defined by*

$$\bar{S}_{m_k,k} := \frac{1}{m_k} \sum_{i=-m_k+1}^{0} F^i x_{k-1}. \tag{6.6}$$

*The average/mean defined in (*6.6*) is also called the* m_k*-local average/mean. Moreover, the* m_k*-local variance corresponding to the local sequence* $S_{m_k,k}$ *in (*6.5*) is defined by*

$$s^2_{m_k,k} := \begin{cases} \frac{1}{m_k} \sum\limits_{i=-m_k+1}^{0} \left(F^i x_{k-1} - \frac{1}{m_k} \sum\limits_{j=-m_k+1}^{0} F^j x_{k-1} \right)^2, & \textit{for small } m_k \\ \frac{1}{m_k-1} \sum\limits_{i=-m_k+1}^{0} \left(F^i x_{k-1} - \frac{1}{m_k} \sum\limits_{j=-m_k+1}^{0} F^j x_{k-1} \right)^2, & \textit{for large } m_k \end{cases} \tag{6.7}$$

DEFINITION 6.2.4 *For each fixed $k \in I(0, N)$, and any $m_k \in I_2(k + r - 1)$, the sequence $\{\bar{S}_{i,k}\}_{i=k-m_k}^{k-1}$ is called a m_k- local moving average/mean process at t_k. Moreover, the sequence $\{s_{i,k}^2\}_{i=k-m_k}^{k-1}$ is called a m_k- local moving variance process at t_k.*

DEFINITION 6.2.5 *Let $\{x(t_i)\}_{i=-r}^{N}$ be a random sample of continuous time stochastic dynamic process collected at partition P in (6.2). The local sample average/mean in (6.6) and local sample variance in (6.7) are called discrete time dynamic processes of sample mean and sample variance statistics.*

DEFINITION 6.2.6 *Let $\{x(t_i)\}_{i=-r}^{N}$ be a random sample of continuous time stochastic dynamic process collected at partition P in (6.2). The m_k-local moving average and variance defined in (6.6) and (6.7) are called the m_k-local moving sample average/mean and local moving sample variance at time t_k, respectively. Moreover, m_k-local sample average and m_k-local sample variance are referred to as local sample mean and local sample variance statistics for the local mean and variance of the continuous time stochastic dynamic system at time t_k, respectively.*

In the following, we derive a dynamic algorithm described by the interconnected discrete-time local conditional sample average/mean and variance dynamic processes. First, we shall state and prove a change in $\bar{S}_{m_k,k}$ and $s_{m_k,k}^2$ with respect to change in time t_k. This fundamental result is motivated by Exercise 5.15 in [14].

DEFINITION 6.2.7 *Let $\{\mathbb{E}[x(t_i)|\mathcal{F}_{t_{i-1}}]\}_{i=-r+1}^{N}$ be a conditional random sample of continuous time stochastic dynamic process with respect to sub-σ algebra $\mathcal{F}_{t_i}$, $t_i \in P$ in (6.2). The m_k-local conditional moving average and variance defined in the context of (6.6) and (6.7) are called the m_k-local conditional moving sample average/mean and local conditional moving sample variance, respectively.*

LEMMA 6.1 ***(Discrete Time Dynamic Model of Local Sample Mean and Sample Variance Process).*** *Let $\{\mathbb{E}[x(t_i)|\mathcal{F}_{t_{i-1}}]\}_{i=-r+1}^{N}$ be a conditional random sample of continuous time stochastic dynamic process with respect to sub-σ algebra $\mathcal{F}_{t_i}$, t_i belong to partition P in (6.2). Let $\bar{S}_{m_k,k}$ and $s_{m_k,k}^2$ be m_k-local conditional sample average and local conditional sample variance at t_k for each $k \in I(0, N)$. Then, an interconnected discrete time dynamic model of local conditional sample mean and sample variance statistics is described by*

$$
\begin{cases}
\bar{S}_{m_{k-p+1},k-p+1} = \frac{m_{k-p}}{m_{k-p+1}}\bar{S}_{m_{k-p},k-p} + \eta_{m_{k-p},k-p}, \quad \bar{S}_{m_0,0} = \bar{S}_0 \\
s^2_{m_k,k} = \begin{cases}
\frac{m_k-1}{m_k}\left[\sum_{i=1}^{p}\left[\frac{m_{k-i}}{\prod_{j=0}^{i-1} m_{k-j}}\right] s^2_{m_{k-i},k-i} + \frac{m_{k-p}}{\prod_{j=0}^{p-1} m_{k-j}}\bar{S}^2_{m_{k-p},k-p}\right] \\
+\varepsilon_{m_{k-1},k-1}, \text{ for small } m_k, m_{k-1} \le m_k \\
\sum_{i=1}^{p}\left[\frac{m_{k-i}-1}{\prod_{j=0}^{i-1} m_{k-j}}\right] s^2_{m_{k-i},k-i} + \frac{m_{k-p}}{\prod_{j=0}^{p-1} m_{k-j}}\bar{S}^2_{m_{k-p},k-p} \\
+\epsilon_{m_{k-1},k-1}, \text{ for large } m_k,\ m_{k-1} \le m_k \\
s^2_{m_i,i} = s^2_i, i \in I_{-p}(0), \quad \text{initial conditions}
\end{cases}
\end{cases}
\tag{6.8}
$$

where

$$
\begin{cases}
\eta_{m_{k-p},k-p} = \frac{1}{m_{k-p+1}}\left[\sum_{i=-m_{k-p+1}+1}^{-m_{k-p}+1} F^i x_{k-p} - F^{-m_{k-p}+1}x_{k-p} - F^{-m_{k-p}}x_{k-p} + F^0 x_{k-p}\right], \\
\varepsilon_{m_{k-1},k-1} = \frac{m_k-1}{m_k}\left[\sum_{i=1}^{p}\frac{\left(F^{-i+1}x_{k-1}\right)^2}{\prod_{j=0}^{i-1} m_{k-j}} - \sum_{i=1}^{p}\frac{\left(F^{-i+1-m_{k-i}}x_{k-1}\right)^2}{\prod_{j=0}^{i-1} m_{k-j}} - \sum_{i=1}^{p}\frac{\left(F^{-i+2-m_{k-i}}x_{k-1}\right)^2}{\prod_{j=0}^{i-1} m_{k-j}}\right] \\
+\frac{m_k-1}{m_k}\left[\sum_{i=1}^{p}\left[\frac{\sum_{l=-i+2-m_{k-i+1}}^{-i+2-m_{k-i}}(F^l x_{k-1})^2}{\prod_{j=0}^{i-1} m_{k-j}}\right]\right. \\
\left.+\sum_{i=1}^{p}\left[\frac{\sum_{\substack{l,s=-i+2-m_{k-i+1} \\ l\neq s}}^{-i+1} F^l x_{k-1}F^s x_{k-1}}{\prod_{j=0}^{i-1} m_{k-j}}\right]\right] - \frac{1}{m_k}\sum_{\substack{l,s=-m_k+1 \\ l\neq s}}^{0} F^l x_{k-1}F^s x_{k-1}, \\
\epsilon_{m_{k-1},k-1} = \sum_{i=1}^{p}\frac{\left(F^{-i+1}x_{k-1}\right)^2}{\prod_{j=0}^{i-1} m_{k-j}} - \sum_{i=1}^{p}\frac{\left(F^{-i+1-m_{k-i}}x_{k-1}\right)^2}{\prod_{j=0}^{i-1} m_{k-j}} - \sum_{i=1}^{p}\frac{\left(F^{-i+2-m_{k-i}}x_{k-1}\right)^2}{\prod_{j=0}^{i-1} m_{k-j}} \\
+\sum_{i=1}^{p}\left[\frac{\sum_{l=-i+2-m_{k-i+1}}^{-i+2-m_{k-i}}(F^l x_{k-1})^2}{\prod_{j=0}^{i-1} m_{k-j}}\right] + \sum_{i=1}^{p}\left[\frac{\sum_{\substack{l,s=-i+2-m_{k-i+1} \\ l\neq s}}^{-i+1} F^l x_{k-1}F^s x_{k-1}}{\prod_{j=0}^{i-1} m_{k-j}}\right] \\
-\frac{1}{m_k-1}\sum_{\substack{l,s=-m_k+1 \\ l\neq s}}^{0} F^l x_{k-1}F^s x_{k-1}
\end{cases}
\tag{6.9}
$$

Proof. The proof of Lemma 6.1 for small m_k, $m_{k-1} \le m_k$, is given in $C.1$. The case for small m_k, $m_k \le m_{k-1}$ is also described in $C.2$. The proof for large m_k, $m_{k-1} \le m_k$, is given in $C.3$. □

REMARK 12 The interconnected system (6.8) can be re-written as the one-step Gauss-Sidel dynamic system [62] of iterative process described by

$$\mathbf{X}(k) = \mathbf{A}(k, \mathbf{X}(k-1))\mathbf{X}(k-1) + \mathbf{e}(k), \tag{6.10}$$

where

$$\begin{aligned}
\mathbf{X}(k) &= \begin{pmatrix} \mathbf{X}_1(k) \\ \mathbf{X}_2(k) \end{pmatrix}, \\
\mathbf{X}_1(k) &= \bar{S}_{m_{k-p+1},k-p+1}, \\
\mathbf{X}_2(k) &= \begin{pmatrix} s^2_{m_{k-p+1},k-p+1} \\ s^2_{m_{k-p+2},k-p+2} \\ \vdots \\ s^2_{m_{k-1},k-1} \\ s^2_{m_k,k} \end{pmatrix}, \\
\mathbf{A}(k, \mathbf{X}(k-1)) &= \begin{pmatrix} \mathbf{A}_{11}(k) & \mathbf{A}_{12}(k) \\ \mathbf{A}_{21}(k, \mathbf{X}(k-1)) & \mathbf{A}_{22}(k) \end{pmatrix}, \\
\mathbf{A}_{11}(k) &= \frac{m_{k-p}}{m_k - p + 1}, \mathbf{A}_{12}(k) = \begin{pmatrix} 0 & 0 & \dots & 0 \end{pmatrix},
\end{aligned}$$

$$\mathbf{A}_{21}(k) = \begin{cases} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \frac{(m_k-1)m_{k-p}}{m_k \prod_{j=0}^{p-1} m_{k-j}} \bar{S}_{m_{k-p},k-p} \end{pmatrix}, & \text{for small } m_k \\ \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \frac{m_{k-p}}{\prod_{j=0}^{p-1} m_{k-j}} \bar{S}_{m_{k-p},k-p} \end{pmatrix}, & \text{for large } m_k, \end{cases},$$

$$
\mathbf{A}_{22}(k) = \begin{cases} \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & 0 & 0 & 0 & \ddots & \vdots \\ 0 & \dots & 0 & 0 & 0 & 1 \\ \frac{(m_k-1)m_{k-p}}{m_k \prod_{j=0}^{p-1} m_{k-j}} & \frac{(m_k-1)m_{k-p+1}}{m_k \prod_{j=0}^{p-2} m_{k-j}} & \dots & \frac{(m_k-1)m_{k-p+i-1}}{m_k \prod_{j=0}^{p-i} m_{k-j}} & \dots & \frac{(m_k-1)m_{k-1}}{m_k^2} \end{pmatrix}, \\ \text{for small } m_k; \\ \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & 0 & 0 & 0 & \ddots & \vdots \\ 0 & \dots & 0 & 0 & 0 & 1 \\ \frac{m_{k-p}-1}{\prod_{j=0}^{p-1} m_{k-j}} & \frac{m_{k-p+1}-1}{\prod_{j=0}^{p-2} m_{k-j}} & \dots & \frac{m_{k-p+i-1}-1}{\prod_{j=0}^{p-i} m_{k-j}} & \dots & \frac{m_{k-1}-1}{m_k} \end{pmatrix}, \text{ for large } m_k \end{cases}
$$

$$
\begin{aligned}
\mathbf{e}(k) &= \begin{pmatrix} \mathbf{e}_1(k) \\ \mathbf{e}_2(k) \end{pmatrix}, \\
\mathbf{e}_1(k) &= \eta(k-p), \\
\mathbf{e}_2(k) &= \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \varepsilon(k-p+1) \end{pmatrix}.
\end{aligned}
$$

REMARK 13 For each $k \in I(0,N)$, $p = 2$ and small m_k, the inter-connected system (6.8) reduces to the following special case:

$$
\begin{cases} \bar{S}_{m_{k-1},k-1} &= \frac{m_{k-2}}{m_{k-1}} \bar{S}_{m_{k-2},k-2} + \eta_{m_{k-2},k-2}, \quad \bar{S}_{m_0,0} = \bar{S}_0 \\ s^2_{m_k,k} &= \frac{m_k-1}{m_k} \left[\frac{m_{k-1}}{m_k} s^2_{m_{k-1},k-1} + \frac{m_{k-2}}{m_k m_{k-1}} s^2_{m_{k-2},k-2} + \frac{m_{k-2}}{m_k m_{k-1}} \bar{S}^2_{m_{k-2},k-2} \right] \\ & \quad + \varepsilon_{m_{k-1},k-1}, \quad s^2_{m_i,i} = s^2_i, \ i \in I_{-2}(0), \end{cases} \tag{6.11}
$$

where

$$\begin{cases} \eta_{m_{k-2},k-2} &= \frac{1}{m_k}\left[\sum_{i=-m_{k-1}+1}^{-m_{k-2}+1} F^i x_{k-2} - F^{-m_{k-2}+1}x_{k-2} - F^{-m_{k-2}}x_{k-2} + F^0 x_{k-2}\right], \\ \varepsilon_{m_{k-1},k-1} &= \frac{m_k-1}{m_k}\left[\frac{\left(F^0 x_{k-1}\right)^2 - \left(F^{-m_{k-1}}x_{k-1}\right)^2 - \left(F^{1-m_{k-1}}x_{k-1}\right)^2}{m_k}\right. \\ & \left. + \frac{\left(F^{-1}x_{k-1}\right)^2 - \left(F^{-1-m_{k-2}}x_{k-1}\right)^2 - \left(F^{-m_{k-2}}x_{k-1}\right)^2}{m_k m_{k-1}}\right] \\ & + \frac{m_k-1}{m_k}\left[\frac{\sum\limits_{i=-m_{k-1}}^{-m_{k-2}} (F^i x_{k-1})^2}{m_k m_{k-1}} + \frac{\sum\limits_{\substack{i,j=-m_{k-1}\\ i\neq j}}^{-1} F^i x_{k-1} F^j x_{k-1}}{m_k m_{k-1}} + \frac{\sum\limits_{i=1-m_k}^{1-m_{k-1}} (F^i x_{k-1})^2}{m_k}\right] \\ & - \frac{\sum\limits_{\substack{i,j=1-m_k\\ i\neq j}}^{0} F^i x_{k-1} F^j x_{k-1}}{m_k^2}. \end{cases}$$

REMARK 14 Define $\varphi_1 = \frac{m_k-1}{m_k}\frac{m_{k-1}}{m_k}$, $\varphi_2 = \frac{m_k-1}{m_k}\frac{m_{k-2}}{m_k m_{k-1}}$, and $\varphi_3 = \frac{m_{k-2}}{m_{k-1}}$. For small m_k, $m_{k-1} \leq m_k$, $\forall k$, we have $\varphi_1 < 1$, $\varphi_2 < 1$, and $\varphi_3 \leq 1$. From $0 < \varphi_i, i = 1, 2, 3$, and the fact that $\varphi_1 + \varphi_2 = \frac{m_k-1}{m_k^2}\left[m_{k-1} + \frac{m_{k-2}}{m_{k-1}}\right] \leq \frac{m_k-1}{m_k^2}\left[m_{k-1}+1\right] \leq \frac{m_k^2-1}{m_k^2} < 1$, the stability of the trivial solution ($X(k) = 0$) of the homogeneous solution corresponding to (6.10) follows. Moreover, under the stated condition, the convergence of solutions of (6.10) also follows.

REMARK 15 Also, (6.11) can be re-written as

$$\mathbf{X}(k) = \mathbf{A}(k, \mathbb{X}(k-1))\mathbf{X}(k-1) + \mathbf{e}(k), \tag{6.12}$$

where $\mathbf{X}(k)$, $\mathbf{A}(k)$ and $\mathbf{e}(k)$ are defined by $\mathbf{X}(k) = \begin{pmatrix}\mathbf{X}_1(k)\\ \mathbf{X}_2(k)\end{pmatrix}$, $\mathbf{X}_1(k) = \bar{S}_{m_{k-1},k-1}$, $\mathbf{X}_2(k) = \begin{pmatrix} s^2_{m_{k-1},k-1} \\ s^2_{m_k,k}\end{pmatrix}$,

$\mathbf{A}(k) = \begin{pmatrix}\mathbf{A}_{11}(k) & \mathbf{A}_{12}(k)\\ \mathbf{A}_{21}(k) & \mathbf{A}_{22}(k)\end{pmatrix}$, $\mathbf{A}_{11}(k) = \frac{m_{k-2}}{m_{k-1}}$, $\mathbf{A}_{12}(k) = \begin{pmatrix}0 & 0\end{pmatrix}$, $\mathbf{A}_{21}(k) = \begin{pmatrix} 0 \\ \frac{(m_k-1)m_{k-2}}{m_k^2 m_{k-1}}\bar{S}_{m_{k-2},k-2}\end{pmatrix}$,

$\mathbf{A}_{22}(k) = \begin{pmatrix} 0 & 1 \\ \frac{(m_k-1)m_{k-2}}{m_k^2 m_{k-1}} & \frac{(m_k-1)m_{k-1}}{m_k^2}\end{pmatrix}$, $\mathbf{e}(k) = \begin{pmatrix}\mathbf{e}_1(k)\\ \mathbf{e}_2(k)\end{pmatrix}$, $\mathbf{e}_1(k) = \eta(k-2)$, $\mathbf{e}_2(k) = \begin{pmatrix} 0 \\ \varepsilon(k-1)\end{pmatrix}$.

REMARK 16 From Remark 13, we note that the local sample variance statistics at time t_k depends on the state of the m_{k-1} and m_{k-2}-local sample variance statistics at time t_{k-1} and t_{k-2}, and the m_{k-2}-local sample mean statistics at time t_{k-2}. We shall later compare the m_k-local sample variance statistics with the GARCH(p,q) model and show that the m_k-local sample variance statistics gives a better forecast than the GARCH(p,q) model under the usage of simulating a real data set.

6.3 Parametric Estimation

In this section, we consider a parameter estimation problem in drift and diffusion coefficients of a very general continuous-time nonlinear stochastic dynamic model described by a systems stochastic differential equations. This problem is motivated by the continuous-time dynamic model validation problem described in (4.11) in the context of energy commodity real data set. This is achieved by utilizing the lagged adaptive process [88] and the interconnected discrete-time dynamics of local sample mean and variances statistic processes model in Section 6.2 (Lemma 6.1). We consider a general system of stochastic differential equations under the influence of hereditary effects in both the drift and diffusion coefficients described by

$$d\mathbf{y} = \mathbf{f}(t, \mathbf{y}_t)dt + \boldsymbol{\sigma}(t, \mathbf{y}_t)d\mathbf{W}(t), \mathbf{y}_{t_0} = \boldsymbol{\varphi}_0, \tag{6.13}$$

where $y_t(\theta) = y(t+\theta)$, $\theta \in [-\tau, 0]$, $\mathbf{f}$, $\boldsymbol{\sigma} : [0,T] \times \mathcal{C} \to \mathbb{R}^q$ are Lipschitz continuous bounded functionals; $\mathcal{C}$ is the Banach space of continuous functions defined on $[-\tau, 0]$ into $\mathbb{R}^q$ equipped with the supremum norm; $W(t)$ is standard Wiener process defined on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathcal{P})$; $\boldsymbol{\varphi}_0 \in \mathcal{C}$, and $\mathbf{y}_0(t_0+\theta)$ is $(\mathcal{F}_{t_0})$ measurables; the filtration function $(\mathcal{F})_{t\geq 0}$ is right-continuous, and each $\mathcal{F}_t$ with $t \geq t_0$ contains all $\mathcal{P}$-null sets in $\mathcal{F}$; the solution process $\mathbf{y}(t_0, \boldsymbol{\varphi}_0)(t)$ of (6.13) is adapted and non-anticipating with respect to $(\mathcal{F}_t)_{t\geq 0}$.

Let $V \in C[[-\tau, \infty] \times \mathbb{R}^q, \mathbb{R}^m]$, and its partial derivatives V_t, $\frac{\partial V}{\partial y}$, $\frac{\partial^2 V}{\partial y^2}$ exist and are continuous. We apply Itô-Doob stochastic differential formula [70] to V , and we obtain

$$dV(t, y) = LV(t, y, y_t)dt + V_y(t, y)\sigma(t, y_t)dW(t) \tag{6.14}$$

where the L operator is defined by

$$\begin{cases} LV(t, y, y_t) &= V_t(t, y) + V_y(t, y)f(t, y_t) + \frac{1}{2}tr(V_{yy}(t, y))b(t, y_t) \\ b(t, y_t) &= \sigma(t, y_t)\sigma^T(t, y_t). \end{cases} \tag{6.15}$$

For (6.13) and (6.14), we present the Euler-type discretization scheme [58]:

$$\begin{cases} \Delta\mathbf{y}_i &= \mathbf{f}(t_{i-1}, \mathbf{y}_{t_{i-1}})\Delta t_i + \boldsymbol{\sigma}(t_{i-1}, \mathbf{y}_{t_{i-1}})\Delta\mathbf{W}_{i-1}, \quad i \in I_1(N) \\ \Delta V(t_i, y(t_i)) &= LV(t_{i-1}, y(t_i), y_{t_{i-1}})\Delta t_i + V_y(t_{i-1}, y(t_{i-1}))\sigma(t_{i-1}, y_{t_{i-1}})\Delta W(t_i) \end{cases} \tag{6.16}$$

Define $\mathcal{F}_{t_{i-1}} \equiv \mathcal{F}_{i-1}$ as the filtration process up to time t_{i-1}. With regard to the continuous time dynamic system (6.13) and its transformed system (6.14), the more general moments of $\Delta\mathbf{y}(t_i)$ are as follows:

$$
\begin{cases}
E\left[\Delta\mathbf{y}(t_i)|\mathcal{F}_{i-1}\right] & = & \mathbf{f}(t_{i-1}, \mathbf{y}_{t_{i-1}})\Delta t_i, \\
E\left[(\Delta\mathbf{y}(t_i) - E\left[\Delta\mathbf{y}(t_i)|\mathcal{F}_{i-1}\right]) \times \right. & & \\
\left. (\Delta\mathbf{y}(t_i) - E\left[\Delta\mathbf{y}(t_i)|\mathcal{F}_{i-1}\right])^T |\mathcal{F}_{i-1}\right] & = & \boldsymbol{\sigma}(t_{i-1}, \mathbf{y}_{t_{i-1}})\boldsymbol{\sigma}^T(t_{i-1}, \mathbf{y}_{t_{i-1}})\Delta t_i, \\
E\left[\Delta V(t_i, y(t_i))|\mathcal{F}_{i-1}\right] & = & LV(t_{i-1}, y(t_i), y_{t_{i-1}})\Delta t_i, \\
E\left[(\Delta V(t_i, y(t_i)) - E\left[\Delta V(t_i, y(t_i))|\mathcal{F}_{i-1}\right]) \times \right. & & \\
\left. (\Delta V(t_i, y(t_i)) - E\left[\Delta V(t_i, y(t_i))|\mathcal{F}_{i-1}\right])^T |\mathcal{F}_{i-1}\right] & = & B(t_{i-1}, y(t_{i-1}), y_{t_{i-1}})
\end{cases}
\tag{6.17}
$$

where $B(t_{i-1}, y(t_{i-1}), y_{t_{i-1}}) = V_y(t_{i-1}, y(t_{i-1}))b(t_{i-1}, y_{t_{i-1}})V_y(t_{i-1}, y(t_{i-1}))^T\Delta t_i$, and T stands for the transpose of the matrix.

From (6.16) and (6.17), we have

$$
\begin{cases}
\Delta\mathbf{y}_i & = & E\left[\Delta\mathbf{y}(t_i)|\mathcal{F}_{i-1}\right] + \boldsymbol{\sigma}(t_{i-1}, \mathbf{y}_{t_{i-1}})\Delta\mathbf{W}_{i-1}, \quad i \in I_1(N) \\
\Delta V(t_i, y(t_i)) & = & E\left[\Delta V(t_i, y(t_i))|\mathcal{F}_{i-1}\right] + V_y(t_{i-1}, y(t_{i-1}))\sigma(t_{i-1}, y_{t_{i-1}})\Delta W(t_i)
\end{cases}
\tag{6.18}
$$

This provides the basis for the development of the concept of lagged adaptive expectation process [88] with respect to continuous time stochastic dynamic systems (6.13) and (6.14). This indeed leads to a formulation of m_k-local generalized method of moments at t_k.

Example 1:

For $V(t, y)$ in (6.14) is defined by $V(t, y) = ||y||_p^p = \sum_{j=1}^{n} |y^j|^p$. In this case, we have

$$
\begin{aligned}
dV &= \left[p\sum_{j=1}^{n} |y^j|^{p-1} sgn(y^j) f(t, y_t^j) + \tfrac{p(p-1)}{2}|y^j|^{p-2}\sigma(t, y_t^j)\right] dt \\
&\quad + p\sum_{j=1}^{n} |y^j|^{p-1} sgn(y^j)\sigma(t, y_t^j) dW^j.
\end{aligned}
\tag{6.19}
$$

Hence, the discretized form of (6.19) is given by

$$
\begin{aligned}
\Delta V_i &= \left[p\sum_{j=1}^{n} |y_{i-1}^j|^{p-1} sgn(y_{i-1}^j) f(t_{i-1}, y_{t_{i-1}}^j) + \tfrac{p(p-1)}{2}|y_{i-1}^j|^{p-2}\sigma(t_{i-1}, y_{t_{i-1}}^j)\right] dt \\
&\quad + p\sum_{j=1}^{n} |y_{i-1}^j|^{p-1} sgn(y_{i-1}^j)\sigma(t_{i-1}, y_{t_{i-1}}^j) dW_i^j.
\end{aligned}
\tag{6.20}
$$

In this special case, (6.18) reduces to

$$\begin{cases} \Delta \mathbf{y}_i & = & E\left[\Delta \mathbf{y}(t_i)|\mathcal{F}_{i-1}\right] + \boldsymbol{\sigma}(t_{i-1}, \mathbf{y}_{t_{i-1}})\Delta \mathbf{W}_{i-1}, \ \ i \in I_1(N) \\ \Delta\left(\sum\limits_{j=1}^{n} |y_i^j|^p\right) & = & E\left[\Delta\left(\sum\limits_{j=1}^{n} |y_i^j|^p\right)|\mathcal{F}_{i-1}\right] + p\sum\limits_{j=1}^{n} |y_{i-1}^j|^{p-1} sgn(y_{i-1}^j)\sigma(t_{i-1}, y_{t_{i-1}}^j)dW_i^j. \end{cases} \tag{6.21}$$

Example 2:

We consider AR(1) model as another example to exhibit the parameter and state estimation problem. The AR(1) model is of the following type

$$X_i = \alpha_{i-1}X_{i-1} + e_i, \ \ X_0 = \mathbf{x}_0, \tag{6.22}$$

where X_i are $\mathcal{F}_i$ measurable, and e_i are independent white noise process and independent of $\mathbf{x}_0$. Hence

$$\begin{cases} E\left[X_i|\mathcal{F}_{i-1}\right] & = & \alpha_{i-1}X_{i-1} \\ E\left[X_iX_i^T|\mathcal{F}_{i-1}\right] & = & \alpha_{i-1}X_{i-1}X_{i-1}^T a_{i-1}^T + E\left[e_ie_i^T|\mathcal{F}_{i-1}\right] \end{cases} \tag{6.23}$$

In the following, we state a result that exhibits the existence of solution of system of non linear equations. For the sake of easy reference, we shall re-state the Implicit function theorem without proof.

THEOREM 6.1 ***Implicit Function Theorem[2]*** *Let $\boldsymbol{F} = \{F_1, F_2, ..., F_q\}$ be a vector-valued function defined on an open set $S \in \mathbb{R}^{q+k}$ with values in $\mathbb{R}^q$. Suppose $\boldsymbol{F} \in \mathcal{C}_1$ on S. Let $(\boldsymbol{u}_0; \boldsymbol{v}_0)$ be a point in S for which $\boldsymbol{F}(\boldsymbol{u}_0; \boldsymbol{v}_0) = 0$ and for which the $q \times q$ determinant det $[D_j\boldsymbol{F}_i(\boldsymbol{u}_0; \boldsymbol{v}_0)] \neq 0$. Then there exists a $k-$ dimensional open set $\boldsymbol{T}_0$ containing $\boldsymbol{v}_0$ and unique vector-valued function $\boldsymbol{g}$, defined on $\boldsymbol{T}_0$ and having values in $\mathbb{R}^q$, such that $\boldsymbol{g} \in \mathcal{C}_1$ on $\boldsymbol{T}_0$, $\boldsymbol{g}(\boldsymbol{v}_0) = \boldsymbol{u}_0$, and $\boldsymbol{F}(\boldsymbol{g}(\boldsymbol{v}); \boldsymbol{v}) = 0$ for every $\boldsymbol{v} \in \boldsymbol{T}_0$.*

6.4 Applications for Illustrations

In the following, we give specific models for different commodities and apply the method of moments to estimate their parameters.

6.4.1 Application 1: Dynamic Model for Energy Commodity Price

We consider a stochastic dynamic model of energy commodities described by the following nonlinear stochastic differential equation

$$dy = ay(\mu - y)dt + \sigma(t, y_t)ydW(t), y_{t_0} = \boldsymbol{\varphi}_0, \tag{6.24}$$

where $y_t(\theta) = y(t+\theta)$; $\theta \in [-\tau, 0]$, $\mu, a \in \mathbb{R}$; the initial process $\boldsymbol{\varphi}_0 = \{y(t_0+\theta)\}_{\theta\in[-\tau,0]}$ is $\mathcal{F}_{t_0}$-measurable and independent of $\{W(t), t \in [0,T]\}$; $W(t)$ is a standard Wiener process defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathcal{P})$ defined in (6.13); $\sigma : [0,T] \times \mathcal{C} \to \mathbb{R}_+$ is a Lipschitz continuous and bounded functional; $\mathcal{C}$ is the Banach space of continuous functions defined on $[-\tau, 0]$ into $\mathbb{R}$ equipped with the supremum norm.

We pick a Lyapunov function $V(t,y) = \ln(y)$ in (6.14) for (6.24). Using Itô-differential formula [70], we have

$$d\left(\ln(y)\right) = \left[a(\mu - y) - \frac{1}{2}\sigma^2(t, y_t)\right]dt + \sigma(t, y_t)dW(t). \tag{6.25}$$

By setting $\Delta t_i = t_i - t_{i-1}$, $\Delta y_i = y_i - y_{i-1}$, the combined Euler discretized scheme for (6.24) and (6.25) is

$$\begin{cases} \Delta y_i &= ay_{i-1}(\mu - y_{i-1})\Delta t_i + \sigma(t_{i-1}, y_{t_{i-1}})y_{i-1}\Delta W(t_i), \quad y_{t_0} = \boldsymbol{\varphi}_0, \\ \Delta\left(\ln(y_i)\right) &= \left[a(\mu - y_{i-1}) - \frac{1}{2}\sigma^2(t_{i-1}, y_{t_{i-1}})\right]\Delta t_i + \sigma(t_{i-1}, y_{t_{i-1}})\Delta W(t_i), y_{t_0} = \boldsymbol{\varphi}_0. \end{cases} \tag{6.26}$$

where $\boldsymbol{\varphi}_0 = \{y_i\}_{i=-r}^{0}$ is a given finite sequence of $\mathcal{F}_0-$ measurable random variables, and it is independent of $\{\Delta W(t_i\}_{i=1}^{N}$.

Applying conditional expectation to (6.26) with respect to $\mathcal{F}_{t_{i-1}} \equiv \mathcal{F}_{i-1}$, we obtain

$$\begin{aligned} \mathbb{E}\left[\Delta y_i | \mathcal{F}_{i-1}\right] &= ay_{i-1}(\mu - y_{i-1})\Delta t \\ \mathbb{E}\left[\Delta\left(\ln(y_i)\right)|\mathcal{F}_{i-1}\right] &= \left[a(\mu - y_{i-1}) - \tfrac{1}{2}\sigma^2(t_{i-1}, y_{t_{i-1}})\right]\Delta t \\ \mathbb{E}\left[\left(\Delta\left(\ln(y_i)\right) - \mathbb{E}\left[\Delta\left(\ln(y_i)\right)|\mathcal{F}_{i-1}\right]\right)^2|\mathcal{F}_{i-1}\right] &= \sigma^2(t_{i-1}, y_{t_{i-1}})\Delta t. \end{aligned} \tag{6.27}$$

From (6.27), (6.26) reduces to

$$\begin{cases} \Delta y_i &= \mathbb{E}\left[\Delta y_i | \mathcal{F}_{i-1}\right] + \sigma(t_{i-1}, y_{t_{i-1}})y_{i-1}\Delta W(t_i) \\ \Delta\left(\ln(y_i)\right) &= \mathbb{E}\left[\Delta\left(\ln(y_i)\right)|\mathcal{F}_{i-1}\right] + \sigma(t_{i-1}, y_{t_{i-1}})\Delta W(t_i). \end{cases} \tag{6.28}$$

(6.28) provides the basis for the development of the concept of lagged adaptive expectation process [88] with respect to continuous time stochastic dynamic systems (6.24) and (6.25).

For $k \in I(0,N)$, applying the lagged adaptive expectation process [88], from Definitions 6.2.3 – 6.2.7, and using (6.8) and (6.28), we formulate a local observation/measurement process at t_k as a algebraic functions of m_k-local functions of restriction of the overall finite sample sequence $\{y_i\}_{i=-r}^{N}$ to subpartition P_k in Definition 6.2.2 :

$$\begin{cases} \frac{1}{m_k}\sum\limits_{i=k-m_k}^{k-1} \mathbb{E}\left[\Delta y_i|\mathcal{F}_{i-1}\right] &= a\left[\frac{\mu}{m_k}\sum\limits_{i=k-m_k}^{k-1} y_{i-1} - \frac{1}{m_k}\sum\limits_{i=k-m_k}^{k-1} y_{i-1}^2\right]\Delta t, \\ \frac{1}{m_k}\sum\limits_{i=k-m_k}^{k-1} \mathbb{E}\left[\Delta\left(\ln(y_i)\right)|\mathcal{F}_{i-1}\right] &= a\left[\mu - \frac{1}{m_k}\sum\limits_{i=k-m_k}^{k-1} y_{i-1}\right]\Delta t \\ & -\frac{1}{2m_k}\sum\limits_{i=k-m_k}^{k-1} \mathbb{E}\left[\left(\Delta\left(\ln(y_i)\right) - \mathbb{E}\left[\Delta\left(\ln(y_i)\right)|\mathcal{F}_{i-1}\right]\right)^2|\mathcal{F}_{i-1}\right], \end{cases} \tag{6.29}$$

$$\hat{\sigma}^2_{m_k,k} = \begin{cases} \frac{1}{m_k\Delta t}\sum\limits_{i=k-m_k}^{k-1} \mathbb{E}\left[\left(\Delta\left(\ln(y_i)\right) - \mathbb{E}\left[\Delta\left(\ln(y_i)\right)|\mathcal{F}_{i-1}\right]\right)^2|\mathcal{F}_{i-1}\right] & \text{if } m_k \text{ is small} \\ \frac{1}{(m_k-1)\Delta t}\sum\limits_{i=k-m_k}^{k-1} \mathbb{E}\left[\left(\Delta\left(\ln(y_i)\right) - \mathbb{E}\left[\Delta\left(\ln(y_i)\right)|\mathcal{F}_{i-1}\right]\right)^2|\mathcal{F}_{i-1}\right] & \text{if } m_k \text{ is large.} \end{cases} \tag{6.30}$$

From (6.30), it follows that the average volatility square $\hat{\sigma}^2_{m_k,k}$ is given by

$$\hat{\sigma}^2_{m_k,k} = \frac{s^2_{m_k,k}}{\Delta t}, \tag{6.31}$$

where $s^2_{m_k,k}$ is the local sample variance statistics for volatility at t_k in the context of $x(t_i) = \Delta\left(\ln(y_i)\right)$. We define

$$\begin{cases} F_1\left(\mathbb{E}\left[\Delta y_i|\mathcal{F}_{i-1}\right], \mathbb{E}\left[\Delta(\ln y_i)|\mathcal{F}_{i-1}\right]; a, \mu\right) &= \frac{\sum\limits_{i=k-m_k}^{k-1} \mathbb{E}[\Delta y_i|\mathcal{F}_{i-1}]}{m_k} \\ & -a\left[\frac{\mu\sum\limits_{i=k-m_k}^{k-1} y_{i-1}}{m_k} - \frac{\sum\limits_{i=k-m_k}^{k-1} y_{i-1}^2}{m_k}\right]\Delta t \\ F_2\left(\mathbb{E}\left[\Delta y_i|\mathcal{F}_{i-1}\right], \mathbb{E}\left[\Delta(\ln y_i)|\mathcal{F}_{i-1}\right]; a, \mu\right) &= \frac{1}{m_k}\sum\limits_{i=k-m_k}^{k-1} \mathbb{E}\left[\Delta(\ln y_i)|\mathcal{F}_{i-1}\right] \\ & -a\left[\mu - \frac{1}{m_k}\sum\limits_{i=k-m_k}^{k-1} y_{i-1}\right]\Delta t + \frac{s^2_{m_k,k}}{2}. \end{cases} \tag{6.32}$$

Then we have

$$\begin{cases} F_1\left(\mathbb{E}\left[\Delta y_i|\mathcal{F}_{i-1}\right], \mathbb{E}\left[\Delta(\ln y_i)|\mathcal{F}_{i-1}\right]; a, \mu\right) &= 0, \\ F_2\left(\mathbb{E}\left[\Delta y_i|\mathcal{F}_{i-1}\right], \mathbb{E}\left[\Delta(\ln y_i)|\mathcal{F}_{i-1}\right]; a, \mu\right) &= 0. \end{cases} \tag{6.33}$$

Let $F = \{F_1, F_2\}$. The determinant of the Jacobian matrix of F is given by

$$\begin{aligned} JF(a,\mu) &= -\tfrac{a}{m_k}\left[\sum_{i=k-m_k}^{k-1} y_{i-1}^2 - \tfrac{1}{m_k}\left(\sum_{i=k-m_k}^{k-1} y_{i-1}\right)^2\right](\Delta t)^2 \\ &= -a\, var(y(t_{i-1})_{i=k-m_k}^{k-1})(\Delta t)^2 \neq 0, \end{aligned} \tag{6.34}$$

provided that $a \neq 0$ or the sequence $\{x(t_{i-1})\}_{i=-r+1}^{N}$ is neither zero nor a constant. This fulfils the hypothesis of Theorem 6.1.

Thus, by the application of Theorem 6.1 (Implicit Function Theorem), we conclude that for every non-constant m_k-local sequence $\{x(t_i)\}_{i=k-m_k}^{k-1}$, there exist a unique solution of system of algebraic equations (6.33), $\hat{a}_{m_k,k}$ and $\hat{\mu}_{m_k,k}$ as a point estimates of a and μ, respectively.

We also note that the estimated values of a and μ change at each time t_k. For instance, at time $t_0 = 0$ and the given $\mathcal{F}_{-1}$ measurable discrete-time process $y_{-r+1}, y_{-r+2}, ..., y_{-1}$, (6.29)-(6.30) reduce to

$$\begin{cases} \frac{1}{m_0}\sum\limits_{i=-m_0}^{0} \Delta y_i &= a\left[\frac{\mu}{m_0}\sum\limits_{i=-m_0}^{0} y_{i-1} - \frac{1}{m_0}\sum\limits_{i=-m_0}^{0} y_{i-1}^2\right]\Delta t, \\ \frac{1}{m_0}\sum\limits_{i=-m_0}^{0} \Delta(\ln y_i) &= a\left[\mu - \frac{1}{m_0}\sum\limits_{i=-m_0}^{0} y_{i-1}\right]\Delta t - \frac{s_{m_0,0}^2}{2}, \\ \hat{\sigma}_{m_0,0}^2 &= \frac{s_{m_0,0}^2}{\Delta t}. \end{cases} \tag{6.35}$$

The initial solution of algebraic equations (6.35) at time t_0 is given by

$$\begin{cases} \hat{a}_{m_0,0} = \dfrac{\left(\frac{1}{m_0}\sum\limits_{i=-m_0}^{0}\Delta(\ln y_i) + \frac{s_{m_0,0}^2}{2}\right)\left(\frac{1}{m_0}\sum\limits_{i=-m_0}^{0} y_{i-1}\right) - \frac{1}{m_0}\sum\limits_{i=-m_0}^{0}\Delta y_i}{\frac{1}{m_0}\left[\sum\limits_{i=-m_0}^{0} y_{i-1}^2 - \frac{1}{m_0}\left(\sum\limits_{i=-m_0}^{0} y_{i-1}\right)^2\right]\Delta t} \\ \hat{\mu}_{m_0,0} = \dfrac{\frac{1}{m_0\Delta t}\sum\limits_{i=-m_0}^{0}\Delta(\ln y_i) + \frac{s_{m_0,0}^2}{2\Delta t} + \frac{\hat{a}_{m_0,0}}{m_0}\left(\sum\limits_{i=-m_0}^{0} y_{i-1}\right)}{\hat{a}_{m_0,0}} \\ \hat{\sigma}_{m_0,0}^2 = \dfrac{s_{m_0,0}^2}{\Delta t}. \end{cases} \tag{6.36}$$

At time $t_1 = 1$ and the given $\mathcal{F}_0$ measurable discrete-time process $y_{-r}, y_{-r+1}, ..., y_{-1}, y_0$, (6.29)-(6.30) reduce to

$$\begin{cases} \frac{1}{m_1}\sum\limits_{i=1-m_1}^{0} \Delta y_i &= a\left[\frac{\mu}{m_1}\sum\limits_{i=1-m_1}^{0} y_{i-1} - \frac{1}{m_1}\sum\limits_{i=1-m_1}^{0} y_{i-1}^2\right]\Delta t, \\ \frac{1}{m_1}\sum\limits_{i=1-m_1}^{0} \Delta(\ln y_i) &= a\left[\mu - \frac{1}{m_1}\sum\limits_{i=1-m_1}^{0} y_{i-1}\right]\Delta t - \frac{s_{m_1,1}^2}{2}, \\ \hat{\sigma}_{m_1,1}^2 &= \frac{s_{m_1,1}^2}{\Delta t}. \end{cases} \tag{6.37}$$

The solution of algebraic equations (6.37) is given by

$$\begin{cases} \hat{a}_{m_1,1} &= \dfrac{\left(\frac{1}{m_1}\sum\limits_{i=1-m_1}^{0}\Delta(\ln y_i)+\frac{s^2_{m_1,1}}{2}\right)\left(\frac{1}{m_1}\sum\limits_{i=1-m_1}^{0}y_{i-1}\right)-\frac{1}{m_1}\sum\limits_{i=1-m_1}^{0}\Delta y_i}{\frac{1}{m_1}\left[\sum\limits_{i=1-m_1}^{0}y^2_{i-1}-\frac{1}{m_1}\left(\sum\limits_{i=1-m_1}^{0}y_{i-1}\right)^2\right]\Delta t} \\ \hat{\mu}_{m_1,1} &= \dfrac{\frac{1}{m_1\Delta t}\sum\limits_{i=1-m_1}^{0}\Delta(\ln y_i)+\frac{s^2_{m_1,1}}{2\Delta t}+\frac{\hat{a}_{m_1,1}}{m_1}\left(\sum\limits_{i=1-m_1}^{0}y_{i-1}\right)}{\hat{a}_{m_1,1}} \\ \hat{\sigma}^2_{m_1,1} &= \dfrac{s^2_{m_1,1}}{\Delta t}. \end{cases} \tag{6.38}$$

Likewise, for $k=2$, we have

$$\begin{cases} \hat{a}_{m_2,2} &= \dfrac{\left(\frac{1}{m_2}\sum\limits_{i=2-m_2}^{1}\Delta(\ln y_i)+\frac{s^2_{m_k,k}}{2}\right)\left(\frac{1}{m_2}\sum\limits_{i=2-m_2}^{1}y_{i-1}\right)-\frac{1}{m_2}\sum\limits_{i=2-m_2}^{1}\Delta y_i}{\frac{1}{m_2}\left[\sum\limits_{i=2-m_2}^{1}y^2_{i-1}+\frac{1}{m_2}\left(\sum\limits_{i=2-m_2}^{1}y_{i-1}\right)^2\right]\Delta t} \\ \hat{\mu}_{m_2,2} &= \dfrac{\frac{1}{m_2\Delta t}\sum\limits_{i=2-m_2}^{1}\Delta(\ln y_i)+\frac{s^2_{m_2,2}}{2\Delta t}+\frac{\hat{a}_{m_2,2}}{m_2}\left(\sum\limits_{i=2-m_2}^{1}y_{i-1}\right)}{\hat{a}_{m_2,2}}, \\ \hat{\sigma}^2_{m_2,2} &= \dfrac{s^2_{m_2,2}}{\Delta t}. \end{cases} \tag{6.39}$$

Hence, from (6.29)-(6.30) and applying the principle of mathematical induction [69], we have

$$\begin{cases} \hat{a}_{m_k,k} &= \dfrac{\left(\frac{1}{m_k}\sum\limits_{i=k-m_k}^{k-1}\Delta(\ln y_i)+\frac{s^2_{m_k,k}}{2}\right)\left(\frac{1}{m_k}\sum\limits_{i=k-m_k}^{k-1}y_{i-1}\right)-\frac{1}{m_k}\sum\limits_{i=k-m_k}^{k-1}\Delta y_i}{\frac{1}{m_k}\left[\sum\limits_{i=k-m_k}^{k-1}y^2_{i-1}-\frac{1}{m_k}\left(\sum\limits_{i=k-m_k}^{k-1}y_{i-1}\right)^2\right]\Delta t} \\ \hat{\mu}_{m_k,k} &= \dfrac{\frac{1}{m_k\Delta t}\sum\limits_{i=k-m_k}^{k-1}\Delta(\ln y_i)+\frac{s^2_{m_k,k}}{2\Delta t}+\frac{\hat{a}_{m_k,k}}{m_k}\left(\sum\limits_{i=k-m_k}^{k-1}y_{i-1}\right)}{\hat{a}_{m_k,k}}, \\ \hat{\sigma}^2_{m_k,k} &= \dfrac{s^2_{m_k,k}}{\Delta t}. \end{cases} \tag{6.40}$$

REMARK 17 We note that without loss in generality, the discrete-time data set $\{y_{-r+i} : i \in I_1(r-1)\}$ is assumed to be close to the true values of the solution process of the continuous-time dynamic process. In fact, this assumption is feasible in view of the uniqueness and continuous dependence of solution process of stochastic functional or ordinary differential equation with respect to the initial data [70].

REMARK 18 If the sample $\{y_i\}_{i=k-m_k-1}^{k-1}$ is a constant sequence, then it follows from (6.40) and the fact that $\Delta(\ln y_i) = 0$ and $s^2_{m_k,k} = 0$, that $\hat{\mu}_{m_k,k} \to \frac{1}{m_k}\sum\limits_{i=k-m_k}^{k-1} y_{i-1}$, and it follows from (6.29)-(6.30) that $\hat{a}_{m_k,k} = 0$.

REMARK 19 As we stated before, estimated parameters a, μ, and σ^2 depend upon the time at which data point is drawn. This is what we expected because of the fact that nonlinearity of the dynamic model generates non stationary solution process. Using this locally estimated parameters of the continuous-time dynamic system, we can find the average of this local parameters over the size of data set as follows:

$$\begin{cases} \bar{a} &= \frac{1}{N}\sum_{i=0}^{N} a_{\hat{m}_i,i}, \\ \bar{\mu} &= \frac{1}{N}\sum_{i=0}^{N} \mu_{\hat{m}_i,i} \\ \overline{\sigma^2} &= \frac{1}{N}\sum_{i=0}^{N} \sigma^2_{\hat{m}_i,i}. \end{cases} \tag{6.41}$$

$\bar{a}$, $\bar{\mu}$, and $\overline{\sigma^2}$ are referred to as aggregated parameter estimates of a, μ, and σ^2 over the given entire finite interval of time, respectively.

6.4.2 Application 2: Dynamic Model for U.S. Treasury Bill Yield Interest Rate and U. S. Eurocurrency Exchange Rate

We also apply the above presented scheme for estimating parameters of a continuous-time model for U.S. Treasury Bill Yield Interest Rate [128] and U. S. Eurocurrency Exchange Rate [129] processes. By employing dynamic modeling process [69, 70], a continuous time dynamic model of interest rate process under random environmental perturbations can be described by

$$dy = (\beta y + \mu y^{\delta})dt + \sigma y^{\gamma} dW(t), y(t_0) = y_0, \tag{6.42}$$

where $\beta, \mu, \delta, \sigma, \gamma \in \mathbb{R}$; $y(t, t_0, y_0)$ is adapted, non-anticipating solution process with respect to $\mathcal{F}_t$; the initial process y_0 is $\mathcal{F}_{t_0}$-measurable and independent of $\{W(t), t \in [t_0, T]\}$; $W(t)$ is a standard Wiener process defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{t\geq 0}, \mathcal{P})$.

For (6.42), we consider the Lyapunov functions $V_1(t, y) = \frac{1}{2}y^2$, and $V_2(t, y) = \frac{1}{3}y^3$ as in (6.14). The It$\hat{o}$ differentials of V_i, for $i = 1, 2$, are given by

$$\begin{cases} dV_1 &= \left[y(\beta y + \mu y^{\delta}) + \frac{1}{2}\sigma^2 y^{2\gamma}\right] dt + \sigma y^{\gamma+1} dW(t) \\ dV_2 &= \left[y^2(\beta y + \mu y^{\delta}) + \sigma^2 y^{2\gamma+1}\right] dt + \sigma y^{\gamma+2} dW(t). \end{cases} \tag{6.43}$$

Following the approach in Section 6.3 and illustration 6.4.1, the Euler discretized scheme ($\Delta t = 1$) for (6.42) is defined by

$$\begin{cases} \Delta y_i &= (\beta y_{i-1} + \mu y_{i-1}^{\delta}) + \sigma y_{i-1}^{\gamma} \Delta W(t_i) \\ \frac{1}{2}\Delta(y_i^2) &= y_{i-1}(\beta y_{i-1} + \mu y_{i-1}^{\delta}) + \frac{1}{2}\sigma^2 y_{i-1}^{2\gamma} + \sigma y_{i-1}^{\gamma+1} \Delta W_i \\ \frac{1}{3}\Delta(y_i^3) &= y_{i-1}^2(\beta y_{i-1} + \mu y_{i-1}^{\delta}) + \sigma^2 y_{i-1}^{2\gamma+1} + \sigma y_{i-1}^{\gamma+2} \Delta W_i. \end{cases} \tag{6.44}$$

Applying conditional expectation to (6.44) with respect to $\mathcal{F}_{i-1}$, we obtain

$$\begin{array}{lcl}
\mathbb{E}\left[\Delta y_i|\mathcal{F}_{i-1}\right] & = & \beta y_{i-1}+\mu y_{i-1}^{\delta} \\
\frac{1}{2}\mathbb{E}\left[\Delta(y_i^2)|\mathcal{F}_{i-1}\right] & = & \beta y_{i-1}^2+\mu y_{i-1}^{\delta+1}+\frac{1}{2}\sigma^2 y_{i-1}^{2\gamma} \\
\frac{1}{3}\mathbb{E}\left[\Delta(y_i^3)|\mathcal{F}_{i-1}\right] & = & \beta y_{i-1}^3+\mu y_{i-1}^{\delta+2}+\sigma^2 y_{i-1}^{2\gamma+1} \\
\mathbb{E}\left[\left(\Delta y_i-\mathbb{E}\left[\Delta y_i|\mathcal{F}_{i-1}\right]\right)^2|\mathcal{F}_{i-1}\right] & = & \sigma^2 y_{i-1}^{2\gamma}, \\
\frac{1}{4}\mathbb{E}\left[\left(\Delta(y_i^2)-\mathbb{E}\left[\Delta(y_i^2)\right]\right)^2|\mathcal{F}_{i-1}\right] & = & \sigma^2 y_{i-1}^{2\gamma+2}.
\end{array} \tag{6.45}$$

From (6.45), (6.44) reduces to

$$\begin{cases}
\Delta y_i & = \mathbb{E}\left[\Delta y_i|\mathcal{F}_{i-1}\right]+\sigma y_{i-1}^{\gamma}\Delta W(t_i) \\
\frac{1}{2}\Delta(y_i^2) & = \frac{1}{2}\mathbb{E}\left[\Delta(y_i^2)|\mathcal{F}_{i-1}\right]+\sigma y_{i-1}^{\gamma+1}\Delta W_i \\
\frac{1}{3}\Delta(y_i^3) & = \frac{1}{3}\mathbb{E}\left[\Delta(y_i^3)|\mathcal{F}_{i-1}\right]+\sigma y_{i-1}^{\gamma+2}\Delta W_i.
\end{cases} \tag{6.46}$$

Following the argument used in (6.29)-(6.30), for $k \in I(0,N)$, applying the lagged adaptive expectation process [88], from Definitions $6.2.3-6.2.7$, and using (6.8) and (6.45), we formulate a local observation/measurement process at t_k as a algebraic functions of m_k-local functions of restriction of the overall finite sample sequence $\{y_i\}_{i=-r}^{N}$ to subpartition P_k in Definition 6.2.2:

$$\begin{cases}
\frac{1}{m_k}\sum_{i=k-m_k}^{k-1}\mathbb{E}\left[\Delta y_i|\mathcal{F}_{i-1}\right] & = \beta\frac{\sum_{i=k-m_k}^{k-1} y_{i-1}}{m_k} +\mu\frac{\sum_{i=k-m_k}^{k-1} y_{i-1}^{\delta}}{m_k} \\
\frac{1}{2m_k}\sum_{i=k-m_k}^{k-1}\left[\mathbb{E}\left[\Delta(y_i^2)|\mathcal{F}_{i-1}\right]-\mathbb{E}\left[\left(\Delta y_i-\mathbb{E}\left[\Delta y_i|\mathcal{F}_{i-1}\right]\right)^2|\mathcal{F}_{i-1}\right]\right] & = \beta\frac{\sum_{i=k-m_k}^{k-1} y_{i-1}^2}{m_k} +\mu\frac{\sum_{i=k-m_k}^{k-1} y_{i-1}^{\delta+1}}{m_k} \\
\frac{1}{m_k}\sum_{i=k-m_k}^{k-1}\left[\frac{1}{3}\mathbb{E}\left[\Delta(y_i^3)|\mathcal{F}_{i-1}\right]-\sigma^2 y_{i-1}^{2\gamma+1}\right] & = \beta\frac{\sum_{i=k-m_k}^{k-1} y_{i-1}^3}{m_k} +\mu\frac{\sum_{i=k-m_k}^{k-1} y_{i-1}^{\delta+2}}{m_k} \\
\frac{1}{m_k}\sum_{i=k-m_k}^{k-1}\mathbb{E}\left[\left(\Delta y_i-\mathbb{E}\left[\Delta y_i|\mathcal{F}_{i-1}\right]\right)^2|\mathcal{F}_{i-1}\right] & = \sigma^2\frac{\sum_{i=k-m_k}^{k-1} y_{i-1}^{2\gamma}}{m_k}, \\
\frac{1}{4m_k}\sum_{i=k-m_k}^{k-1}\mathbb{E}\left[\left(\Delta(y_i^2)-\mathbb{E}\left[\Delta(y_i^2)\right]\right)^2|\mathcal{F}_{i-1}\right] & = \sigma^2\frac{\sum_{i=k-m_k}^{k-1} y_{i-1}^{2\gamma+2}}{m_k}.
\end{cases} \tag{6.47}$$

Following the approach discussed in Section 6.4.1, the solution of $\sigma_{m_k,k}$ is given by

$$\sigma_{m_k,k} = \left[\frac{s^2_{m_k,k}}{\frac{1}{m_k}\sum\limits_{i=k-m_k}^{k-1} y_{i-1}^{2\gamma_{m_k,k}}} \right]^{1/2} \tag{6.48}$$

and $\gamma_{m_k,k}$ satisfies the following nonlinear algebraic equation

$$s^2_{m_k,k}\sum_{i=k-m_k}^{k-1} y_{i-1}^{2\gamma_{m_k,k}+2} - \frac{1}{4}\mathbf{s}^2_{m_k,k}\sum_{i=k-m_k}^{k-1} y_{i-1}^{2\gamma_{m_k,k}} = 0, \tag{6.49}$$

where $s^2_{m_k,k}$ and $\mathbf{s}^2_{m_k,k}$ denotes the local moving variance of Δy_i and $\Delta(y_i^2)$ respectively.

To solve for the parameters β, μ and δ, we define the conditional moment functions

$$F_j \equiv F_j\left(\mathbb{E}\left[\Delta y_i|\mathcal{F}_{i-1}\right], \mathbb{E}\left[\Delta(y_i)^2|\mathcal{F}_{i-1}\right], \mathbb{E}\left[\Delta(y_i)^3|\mathcal{F}_{i-1}\right]\right), \quad j = 1,2,3$$

as

$$\begin{cases} F_1 = \frac{1}{m_k}\sum\limits_{i=k-m_k}^{k-1}\mathbb{E}\left[\Delta y_i|\mathcal{F}_{i-1}\right] - \beta\frac{\sum\limits_{i=k-m_k}^{k-1} y_{i-1}}{m_k} - \mu\frac{\sum\limits_{i=k-m_k}^{k-1} y_{i-1}^{\delta}}{m_k} \\ F_2 = \frac{1}{2m_k}\sum\limits_{i=k-m_k}^{k-1}\left[\mathbb{E}\left[\Delta(y_i^2)|\mathcal{F}_{i-1}\right] - \mathbb{E}\left[\left(\Delta y_i - \mathbb{E}\left[\Delta y_i|\mathcal{F}_{i-1}\right]\right)^2|\mathcal{F}_{i-1}\right]\right] - \beta\frac{\sum\limits_{i=k-m_k}^{k-1} y_{i-1}^2}{m_k} \\ \quad -\mu\frac{\sum\limits_{i=k-m_k}^{k-1} y_{i-1}^{\delta+1}}{m_k} \\ F_3 = \frac{1}{m_k}\sum\limits_{i=k-m_k}^{k-1}\left[\frac{1}{3}\mathbb{E}\left[\Delta(y_i^3)|\mathcal{F}_{i-1}\right] - \sigma^2 y_{i-1}^{2\gamma+1}\right] - \beta\frac{\sum\limits_{i=k-m_k}^{k-1} y_{i-1}^3}{m_k} - \mu\frac{\sum\limits_{i=k-m_k}^{k-1} y_{i-1}^{\delta+2}}{m_k}. \end{cases} \tag{6.50}$$

Using (6.47), we have

$$\begin{cases} F_1 = 0 \\ F_2 = 0 \\ F_3 = 0 \end{cases} \tag{6.51}$$

Let $F = \{F_1, F_2, F_3\}$. The determinant of the Jacobian matrix of F is given by

$$JF(\beta,\mu,\delta) = -\frac{1}{m_k^3} det \begin{pmatrix} \sum\limits_{i=k-m_k}^{k-1} y_{i-1} & \sum\limits_{i=k-m_k}^{k-1} y_{i-1}^{\delta} & \sum\limits_{i=k-m_k}^{k-1} (\ln y_{i-1}) y_{i-1}^{\delta} \\ \sum\limits_{i=k-m_k}^{k-1} y_{i-1}^2 & \sum\limits_{i=k-m_k}^{k-1} y_{i-1}^{\delta+1} & \sum\limits_{i=k-m_k}^{k-1} (\ln y_{i-1}) y_{i-1}^{\delta+1} \\ \sum\limits_{i=k-m_k}^{k-1} y_{i-1}^3 & \sum\limits_{i=k-m_k}^{k-1} y_{i-1}^{\delta+2} & \sum\limits_{i=k-m_k}^{k-1} (\ln y_{i-1}) y_{i-1}^{\delta+2} \end{pmatrix} \neq 0 \tag{6.52}$$

provided $\delta \neq 1$ and the sequence $\{y(t_{i-1})\}_{i=k-m_k}^{k-1}$ is neither zero nor a constant. We want to avoid the case where $\delta = 1$ because this will change the structure of (6.42). Thus, by the application of Theorem 6.1 (Implicit Function Theorem), we conclude that for every non-constant m_k-local sequence $\{y(t_i)\}_{i=k-m_k}^{k-1}$, $\delta \neq 1$, there exist a solution of system of algebraic equations (6.51) $\hat{\beta}_{m_k,k}$, $\hat{\mu}_{m_k,k-1}$, $\hat{\delta}_{m_k,k}$ as a point estimates of β and μ, and δ respectively.

The solution of (6.51) is given by

$$\begin{cases} \hat{\mu}_{m_k,k} = \dfrac{\frac{1}{m_k}\sum\limits_{i=k-m_k}^{k-1}\Delta y_i \sum\limits_{i=k-m_k}^{k-1} y_{i-1}^2 - \frac{1}{2}\left[\frac{1}{m_k}\sum\limits_{i=k-m_k}^{k-1}\Delta(y_i^2) - s_{m_k,k}^2\right]\sum\limits_{i=k-m_k}^{k-1} y_{i-1}}{\frac{1}{m_k}\left[\sum\limits_{i=k-m_k}^{k-1} y_{i-1}^{\delta_{m_k,k}} \sum\limits_{i=k-m_k}^{k-1} y_{i-1}^2 - \sum\limits_{i=k-m_k}^{k-1} y_{i-1}^{1+\delta_{m_k,k}} \sum\limits_{i=k-m_k}^{k-1} y_{i-1}\right]} \\ \hat{\beta}_{m_k,k} = \dfrac{\sum\limits_{i=k-m_k}^{k-1}\Delta y_i - \hat{\mu}_{m_k,k}\sum\limits_{i=k-m_k}^{k-1} y_{i-1}^{\delta_{m_k,k}}}{\sum\limits_{i=k-m_k}^{k-1} y_{i-1}}, \end{cases} \tag{6.53}$$

where $\delta_{m_k,k}$ satisfies the third equation in (6.47) described by

$$\frac{1}{3m_k}\sum_{i=k-m_k}^{k-1}\Delta(y_i^3) - \frac{\sigma_{m_k,k}^2}{m_k}\sum_{i=k-m_k}^{k-1} y_{i-1}^{2\gamma_{m_k,k}+1} - \beta\frac{\sum\limits_{i=k-m_k}^{k-1} y_{i-1}^3}{m_k} - \mu\frac{\sum\limits_{i=k-m_k}^{k-1} y_{i-1}^{\delta+2}}{m_k} = 0 \tag{6.54}$$

Chapter 7

Computational and Simulation Algorithms

7.1 Introduction

In this chapter, we outline computational, data organizational and simulation schemes. We introduce the ideas of iterative data process and data simulation time schedules in relation with the real time data observation/collection schedule. For the computational estimation of continuous time stochastic dynamic system state and parameters, it is essential to identify an admissible set of local conditional sample average and sample variance parameters, namely, the size of local conditional sample in the context of a partition of time interval $[-\tau, T]$. Moreover, the discrete time dynamic model of conditional sample mean and sample variance statistics processes in Section 6.2 and the theoretical parameter estimation scheme in Section 6.3 motivates to outline a computational scheme in a systematic and coherent manner. A brief conceptual computational scheme and simulation process summary is described below:

7.2 Coordination of Data Observation, Iterative Process, and Simulation Schedules:

Without loss of generality, we assume that the real data observation/collection partition schedule P is defined in (6.2). Now, we present definitions of iterative process and simulation time schedule.

DEFINITION 7.2.1 *The iterative process time schedule in relation with the real data collection schedule is defined by*

$$IP = \{F^{-r}t_i : \ \textit{for} \ t_i \in P\}, \tag{7.1}$$

where $F^{-r}t_i = t_{i-r}$, and F^{-r} is a forward shift operator [11].

The simulation time is based on the order p of the time series model of m_k-local conditional sample mean and variance processes in Lemma 6.1.

DEFINITION 7.2.2 *The simulation process time schedule in relation with the real data observation schedule is defined by*

$$SP = \begin{cases} \{F^r t_i : \ \textit{for} \ t_i \in P\}, \ \textit{if} \ p \le r \\ \{F^p t_i : \ \textit{for} \ t_i \in P\}, \ \textit{if} \ p > r. \end{cases} \tag{7.2}$$

REMARK 20 We note that the initial times of iterative and simulation processes are equal to the real data times t_r and t_p, respectively. Moreover, iterative and simulation processes time in (7.1) and (7.2), respectively justify Remark 17. In short, t_i is the scheduled time clock for the collection of the ith observation of the state of the system under investigation. The iterative process and simulation process times are t_{i+r} and t_{i+p}, respectively.

7.3 Conceptual Computational Parameter Estimation Scheme

For the conceptual computational dynamic system parameter estimation, we need to introduce a few concepts of local admissible sample/data observation size, m_k-local admissible conditional finite sequence at $t_k \in SP$, local finite sequence of parameter estimates at t_k.

DEFINITION 7.3.1 *For each $k \in I(0, N)$, we define local admissible sample/data observation size m_k at t_k as $m_k \in OS_k$, where*

$$OS_k = \begin{cases} I(2, r + k - 1), \ \textit{if}\, p \le r, \\ I(2, p + k - 1), \ \textit{if}\, p > r, \end{cases} \tag{7.3}$$

Moreover, OS_k is referred as the local admissible set of lagged sample/data observation size at t_k.

DEFINITION 7.3.2 *For each admissible $m_k \in OS_k$ in Definition 7.3.1, a m_k-local admissible lagged-adapted finite restriction sequence of conditional sample/data observation at t_k to subpartition P_k of P in Definition 6.2.3 is defined by $\{\mathbb{E}[y_i|\mathcal{F}_{i-1}]\}_{i=k-m_k}^{k-1}$. Moreover, a m_k- class of admissible lagged-adapted finite sequences of conditional sample/data observation of size m_k at t_k is defined by*

$$\mathcal{AS}_k = \{\{\mathbb{E}[y_i|\mathcal{F}_{i-1}]\}_{i=k-m_k}^{k-1} : \ m_k \in OS_k\} = \{\{\mathbb{E}[y_i|\mathcal{F}_{i-1}]\}_{i=k-m_k}^{k-1}\}_{m_k \in OS_k}. \tag{7.4}$$

In the case of energy commodity model, for each $m_k \in OS_k$, we find corresponding m_k- local admissible adapted finite sequence of conditional sample/data observation at t_k, $\{\mathbb{E}[y_i|\mathcal{F}_{i-1}]\}_{i=k-m_k}^{k-1}$. Using this sequence and (6.40), we compute $\hat{a}_{m_k,k}$, $\hat{\mu}_{m_k,k}$ and $\hat{\sigma}^2_{m_k,k}$. This leads to a local finite sequence of parameter estimates at t_k defined on OS_k as follows: $\left\{(\hat{a}_{m_k,k}, \hat{\mu}_{m_k,k}, \hat{\sigma}^2_{m_k,k})\right\}_{m_k \in OS_k} =$

$\{(\hat{a}_{m_k,k}, \hat{\mu}_{m_k,k}, \hat{\sigma}^2_{m_k,k})\}_{m_k \in 2}^{r+k-1}$ *or* $\{(\hat{a}_{m_k,k}, \hat{\mu}_{m_k,k}, \hat{\sigma}^2_{m_k,k})\}_{m_k \in 2}^{p+k-1}$.
It is denoted by $(\mathcal{A}_k, \mathbb{M}_k, \mathcal{S}_k)$=$\Big\{(\hat{a}_{m_k,k}, \hat{\mu}_{m_k,k}, \hat{\sigma}^2_{m_k,k})\Big\}_{m_k \in OS_k}$.

7.4 Conceptual Computation of State Simulation Scheme

For the development of a conceptual computational scheme, we need to employ the method of induction. The presented simulation scheme is based on the idea of lagged adaptive expectation process [88]. An autocorrelation function (ACF) analysis [11, 14] performed on $s^2_{m_k,k}$ suggests that the interconnected discrete time dynamic model of local conditional sample mean and sample variance statistics in (6.8) is of order $p = 2$. In view of this, we need to identify the initial data. We begin with a given initial data y_{t_0}, $\{\hat{s}^2_{m_0,0}\}_{m_0 \in OS_0}$, $\{\hat{s}^2_{m_{-1},-1}\}_{m_{-1} \in OS_{-1}}$, $\{\bar{S}^2_{m_{-1},-1}\}_{m_{-1} \in OS_{-1}}$. Let $y^s_{m_k,k}$ be a simulated value of $\mathbb{E}[y_k|\mathcal{F}_{k-1}]$ at time t_k corresponding to an admissible sequence $\{\mathbb{E}[y_i|\mathcal{F}_{i-1}]\}_{i=k-m_k}^{k-1} \in \mathcal{AS}_k$. This simulated value is derived from the discretized Euler scheme (6.26) by

$$y^s_{m_k,k} = y^s_{m_{k-1},k-1} + \hat{a}_{m_{k-1},k-1}(\hat{\mu}_{m_{k-1},k-1} - y^s_{m_{k-1},k-1}) y^s_{m_{k-1},k-1}\Delta t + \hat{\sigma}_{m_{k-1},k-1} y^s_{m_{k-1},k-1}\Delta W_{m_k,k}. \tag{7.5}$$

Let $\{y^s_{m_k,k}\}_{m_k \in OS_k}$ be a m_k- local sequence of simulated values corresponding to m_k-admissible lagged adapted finite sequence of conditional observation belonging to $\mathcal{AS}_k$ and corresponding term of sequence $(\mathcal{A}_k, \mathbb{M}_k, \mathcal{S}_k)$. Thus, $\{y^s_{m_k,k}\}_{m_k \in OS_k}$ is the finite sequence corresponding to finite simulated values of $\mathbb{E}[y_k|\mathcal{F}_{k-1}]$ at t_k.

7.5 Mean-Square Sub-Optimal Procedure

To find the the best estimate of $\mathbb{E}[y_k|\mathcal{F}_{k-1}]$ using a local admissible finite sequence $\{y^s_{m_k,k}\}_{m_k \in OS_k}$ of simulation of $\{\mathbb{E}[y_i|\mathcal{F}_{i-1}]\}$, we need to compute a finite sequence of quadratic mean square error corresponding to $\{y^s_{m_k,k}\}_{m_k \in OS_k}$. The quadratic mean square error is defined below.

DEFINITION 7.5.1 *The quadratic mean square error of* $\mathbb{E}[y_k|\mathcal{F}_{k-1}]$ *relative to each member of the term of local admissible sequence* $\{y^s_{m_k,k}\}_{m_k \in OS_k}$ *of simulated values is defined by*

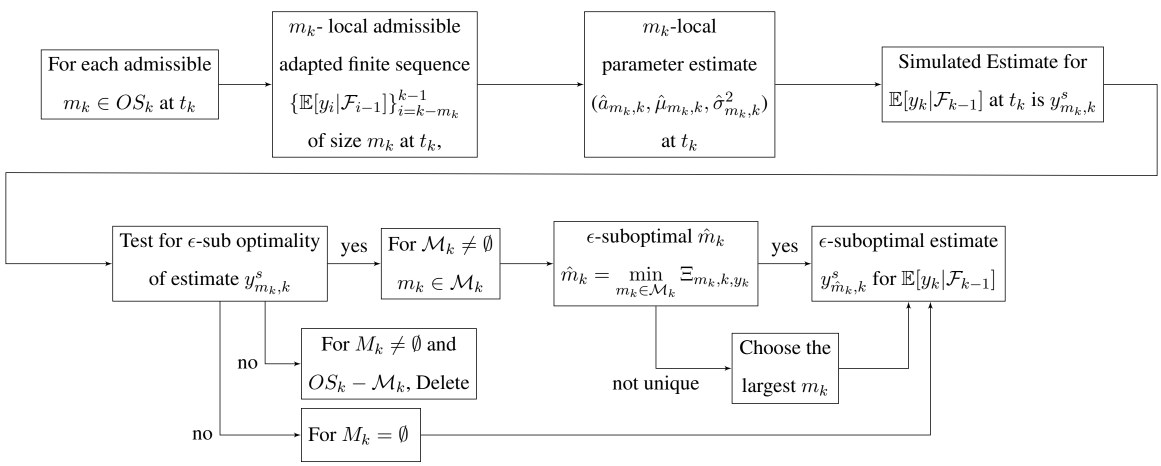
$$\Xi_{m_k,k,y_k} = \left(\mathbb{E}[y_k|\mathcal{F}_{k-1}] - y^s_{m_k,k}\right)^2. \tag{7.6}$$

For any arbitrary small positive number ϵ and for each time t_k, to find the the best estimate from the admissible simulated values of simulated sequence of $\{y^s_{m_k,k}\}_{m_k \in OS_k}$ for $\mathbb{E}[y_k|\mathcal{F}_{k-1}]$, we determine the following sub-optimal admissible set of m_k-size local conditional sample

$$\mathcal{M}_k = \{m_k : \Xi_{m_k,k,y_k} < \epsilon \text{ for } m_k \in OS_k\}. \tag{7.7}$$

Among these collected values, the value that gives the minimum Ξ_{m_k,k,y_k} is recorded as $\hat{m}_k$. If more than one value of m_k exists, then the largest of such m_k's is recorded as $\hat{m}_k$. If condition (7.7) is not met at time t_k, the value of m_k where the minimum $\min\limits_{m_k} \Xi_{m_k,k,y_k}$ is attained, is recorded as $\hat{m}_k$. The $\epsilon-$ level sub-optimal estimates of the parameters $\hat{a}_{m_k,k}$, $\hat{\mu}_{m_k,k}$ and $\hat{\sigma}^2_{m_k,k}$ at $\hat{m}_k$ are also recorded as $a_{\hat{m}_k,k}$, $\mu_{\hat{m}_k,k}$ and $\sigma^2_{\hat{m}_k,k}$, respectively. Finally, the simulated value $y^s_{m_k,k}$ at time t_k with $\hat{m}_k$ is now recorded as the best estimate for $\mathbb{E}[y_k|\mathcal{F}_{k-1}]$ at t_k. This value is called the $\epsilon-$ sub-optimal simulated value $y^s_{\hat{m}_k,k}$ of $\mathbb{E}[y_k|\mathcal{F}_{k-1}]$ at t_k. Similar reasoning can be provided for the estimates of the parameters of the U.S. Treasury Bill Yield Interest Rate and U. S. Eurocurrency Exchange Rate model. A detailed flowchart of the conceptual algorithm is as follows:

Flowchart 1: LLGMM Conceptual Computational Algorithm.



Moreover, a detailed simulation algorithm is presented in $C.4$

7.6 Applications: Four Energy Commodity Data Sets

Now, we apply the above conceptual computational algorithm for the real time data sets namely daily Henry Hub Natural gas data set for the period $01/04/2000$-$09/30/2004$, daily crude oil data set for the period $01/07/1997 - 06/02/2008$, daily coal data set for the period of $01/03/2000 - 10/25/2013$, and weekly ethanol data set for the period of $03/24/2005 - 09/26/2013$, [22, 24, 23, 136]. Using $\Delta t = 1$, $\epsilon = 0.001$, $r = 5$, and $p = 2$, the $\epsilon-$ level sub-optimal estimates of parameters a, μ and σ^2 at each real data times are exhibited in Table 6.

Table 6: Estimates $\hat{m}_k$, $\sigma^2_{\hat{m}_k,k}$, $\mu_{\hat{m}_k,k}$ and $a_{\hat{m}_k,k}$.

t_k	Natural gas				t_k	Crude oil				t_k	Coal				t_k	Ethanol			
	$\hat{m}_k$	$\sigma^2_{\hat{m}_k,k}$	$\mu_{\hat{m}_k,k}$	$a_{\hat{m}_k,k}$		$\hat{m}_k$	$\sigma^2_{\hat{m}_k,k}$	$\mu_{\hat{m}_k,k}$	$a_{\hat{m}_k,k}$		$\hat{m}_k$	$\sigma^2_{\hat{m}_k,k}$	$\mu_{\hat{m}_k,k}$	$a_{\hat{m}_k,k}$		$\hat{m}_k$	$\sigma^2_{\hat{m}_k,k}$	$\mu_{\hat{m}_k,k}$	$a_{\hat{m}_k,k}$
5	3	0.0001	2.2231	0.6011	5	3	0.0001	24.4100	0.0321	5	3	0.0001	11.5534	0.0142	5	2	0.0002	1.1767	0.5831
6	3	0.0002	2.2160	0.6122	6	3	0.0002	24.7165	0.0341	6	3	0.0000	11.2529	0.4109	6	5	0.0008	1.1717	0.5159
7	3	0.0002	2.2513	0.6087	7	4	0.0003	25.5946	0.0537	7	3	0.0001	9.9161	0.0165	7	4	0.0007	1.1707	1.4925
8	4	0.0002	2.2494	0.1628	8	5	0.0006	25.5550	0.0467	8	3	0.0002	11.4663	-0.0403	8	5	0.0008	1.1713	1.4791
9	4	0.0002	2.2658	-0.1497	9	4	0.0006	25.5695	0.0499	9	3	0.0005	10.5922	-0.0843	9	5	0.0006	1.1709	2.1406
10	4	0.0003	2.1371	0.1968	10	4	0.0004	25.4787	0.0221	10	4	0.0009	8.9379	0.0714	10	4	0.0004	1.1900	0.8621
11	4	0.0004	2.5071	-0.2781	11	3	0.0001	25.7742	0.0100	11	4	0.0023	8.9051	0.1784	11	3	0.0025	1.1900	0.3719
12	4	0.0000	2.2550	0.3545	12	3	0.0002	26.9477	-0.0157	12	3	0.0015	9.0169	0.0855	12	3	0.0004	1.2188	0.5368
13	4	0.0005	2.5122	0.6246	13	3	0.0001	25.8786	-0.0112	13	3	0.0020	8.6231	0.0739	13	5	0.0004	1.1120	12.2917
14	4	0.0015	2.4850	0.5604	14	5	0.0005	22.1834	0.0049	14	2	0.0001	10.0100	0.0564	14	5	0.0007	1.1669	-0.9289
15	3	0.0007	2.5378	0.4846	15	5	0.0004	23.5425	0.0010	15	5	0.0067	9.5281	0.0741	15	5	0.0014	0.7492	-0.0879
16	3	0.0007	2.5715	0.7737	16	4	0.0002	23.8500	0.0000	16	4	0.0058	6.1821	0.0694	16	5	0.0011	1.7968	0.3087
17	5	0.0011	2.5688	0.5984	17	4	0.0002	23.8486	0.0502	17	4	0.0015	8.8087	0.0404	17	5	0.0002	1.8484	-0.1901
18	4	0.0010	2.5831	0.5423	18	5	0.0004	23.2913	-0.0113	18	4	0.0035	9.0681	0.0652	18	5	0.0003	1.1650	-0.1611
19	5	0.0007	2.5893	0.4256	19	3	0.0000	24.4715	0.1282	19	3	0.0040	9.0752	0.1527	19	5	0.0022	1.8943	0.1502
20	5	0.0006	2.6100	0.0683	20	3	0.0004	24.3878	0.0415	20	3	0.0049	9.0801	0.1405	20	5	0.0047	1.8144	0.2073
...	...	...	..	...	...	...	...	..	...	...	...	...	..	..	...	...	...	...	...
...	...	...	..	...	...	...	...	..	...	...	...	...	..	..	...	...	...	...	...
781	3	0.0001	6.2830	0.0307	2446	4	0.0001	59.6591	0.0012	2871	5	0.0008	26.9028	0.0131	281	3	0.0003	1.6026	-1.8731
782	4	0.0004	5.6052	0.0450	2447	5	0.0007	59.5048	0.0059	2872	5	0.0008	26.7876	0.0158	282	5	0.0007	1.7479	0.2390
783	4	0.0009	5.9775	0.0996	2448	4	0.0005	58.9636	0.0050	2873	4	0.0003	26.6849	-0.0313	283	4	0.0010	3.7451	0.0171
784	4	0.0230	13.6910	0.3781	2449	5	0.0004	58.4700	0.0042	2874	5	0.0008	25.7722	0.0231	284	5	0.0008	1.9909	0.0267
785	5	0.0189	9.7555	-0.1031	2450	4	0.0002	58.5012	0.0071	2875	4	0.0009	25.5993	0.0253	285	5	0.0005	1.8650	0.5265
786	5	0.0213	9.2861	0.0550	2451	4	0.0003	59.2442	0.0048	2876	5	0.0007	25.6284	0.0544	286	5	0.0016	1.8887	1.1808
787	5	0.0149	9.2482	0.0501	2452	4	0.0003	59.0349	-0.0 015	2877	4	0.0006	25.2403	0.0364	287	5	0.0014	1.2920	0.1535
788	5	0.0083	9.1330	0.0985	2453	5	0.0004	59.2438	0.0006	2878	5	0.0009	25.3519	0.0301	288	5	0.0006	2.0086	0.5159
789	5	0.0161	6.5570	0.2080	2454	5	0.0004	60.1814	0.0033	2879	4	0.0005	25.0336	0.0528	289	4	0.0042	1.9985	0.4236
790	5	0.0114	7.0045	0.0989	2455	4	0.0003	59.9567	0.0091	2880	5	0.0006	25.2560	0.0963	290	4	0.0030	1.9857	0.5156
791	5	0.0071	0.5917	0.1034	2456	5	0.0004	59.6785	0.0025	2881	4	0.0003	25.3842	0.0438	291	4	0.0023	2.0219	0.8551
792	5	0.0096	6.9603	0.1118	2457	4	0.0004	58.7682	0.0049	2882	3	0.0006	25.5547	0.0552	292	5	0.0030	2.2320	0.2054
793	5	0.0045	6.8894	0.0341	2458	4	0.0003	58.4793	0.0086	2883	3	0.0005	25.5867	0.0719	293	4	0.0009	2.2062	0.0655
794	5	0.0056	7.3016	0.2594	2459	4	0.0001	57.9168	0.0053	2884	5	0.0008	25.5035	0.0649	294	3	0.0009	2.3554	0.0893
795	5	0.0049	3.9781	0.3501	2460	4	0.0004	56.1216	0.0177	2885	3	0.0004	25.5128	0.0690	295	3	0.0018	2.3115	0.5336

Table 6 shows the estimates of the ϵ- sub-optimal size $\hat{m}_k$, the parameters $\sigma^2_{\hat{m}_k,k}$, $\mu_{\hat{m}_k,k}$ and $a_{\hat{m}_k,k}$ for each of the energy commodity data sets. Moreover, $p \leq r$, and the initial real data time is $t_r = t_5$.

In the following, the graph of $a_{\hat{m}_k,k}$ for daily natural gas, daily crude oil, daily coal, and weekly ethanol are exhibited in Figure 8 (a), (b), (c) and (d), respectively.

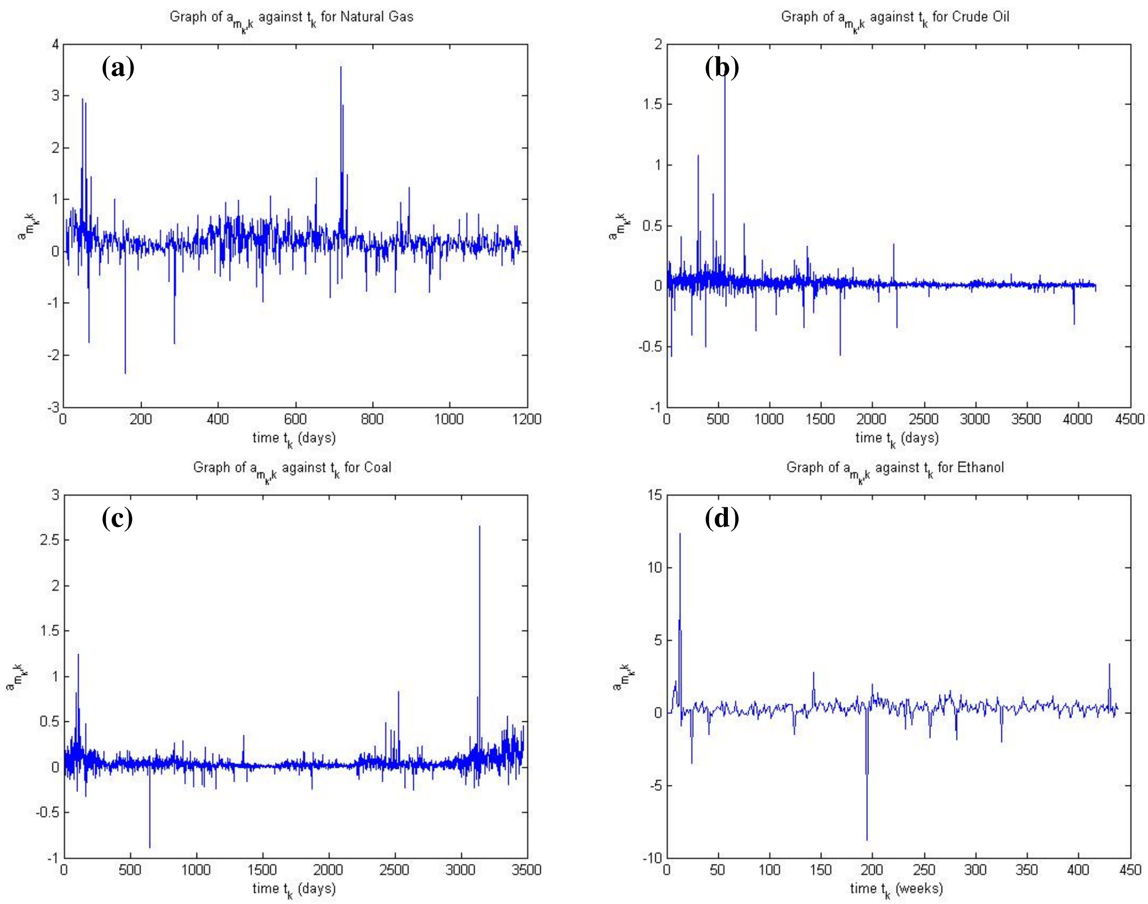


Figure 8.: The graph of mean reverting rate $a_{\hat{m}_k,k}$ with time t_k

Figures 8: **(a), (b), (c)and (d)** are the graphs of $a_{\hat{m}_k,k}$ against time t_k for the daily Henry Hub natural gas data set [24] , daily crude oil data set [23], daily coal data set [22], and weekly ethanol data set [136], respectively. It shows the rate at which the data sets are reverting to the mean level.

Furthermore, we show the graphs of $\mu_{\hat{m}_k,k}$ for each of the data set: Natural gas, Crude Oil, Coal and Ethanol in Figure 9 (a), (b), (c) and (d), respectively.

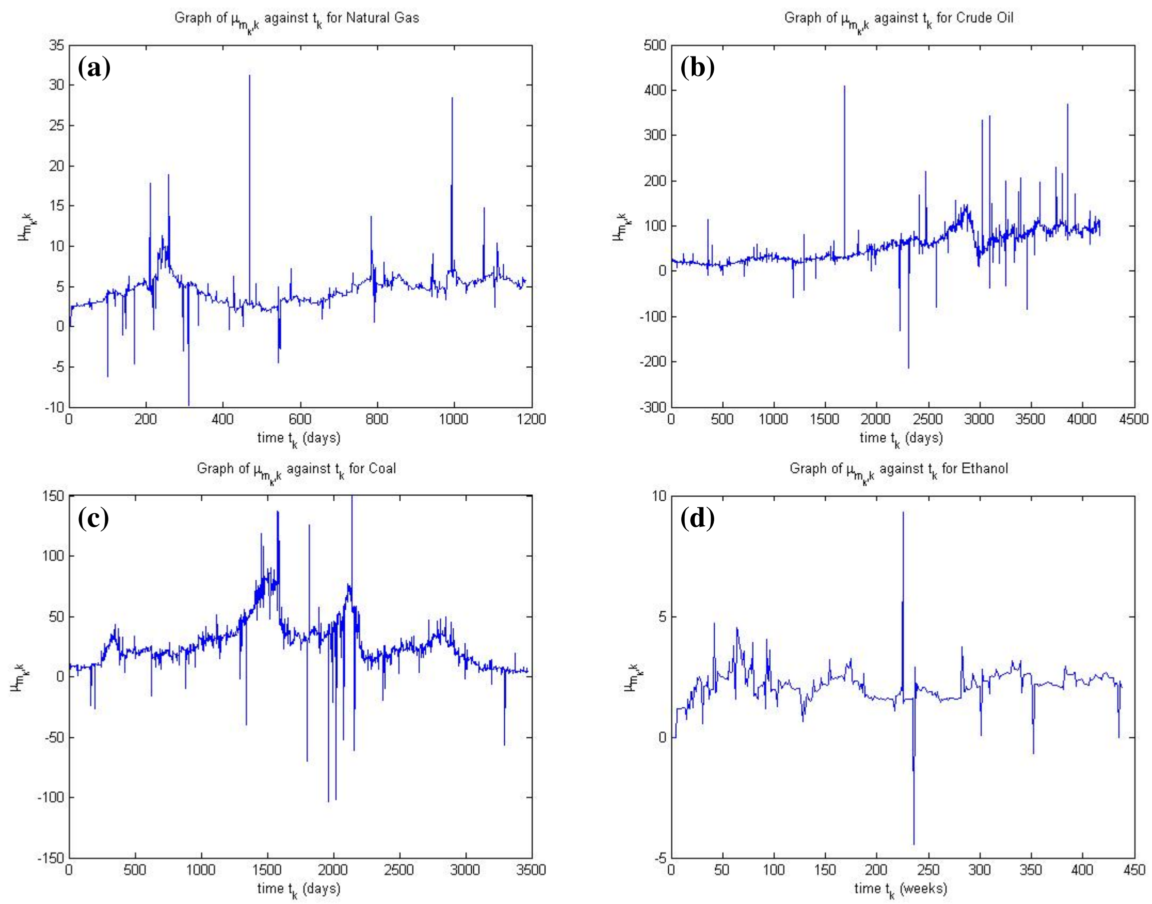


Figure 9.: The graph of mean level $\mu_{\hat{m}_k,k}$ with time t_k

Figures 9: **(a), (b), (c)and (d)** are the graphs of $\mu_{\hat{m}_k,k}$ against time t_k for the daily Henry Hub natural gas data set [24], daily crude oil data set [23], daily coal data set [22], and weekly ethanol data set [136] respectively. The sample means of the real data y_k sets for Natural gas, Crude oil data and Coal data are given by 4.5385, 54.0093, 27.1441 and 2.1391 respectively. It can be seen from Figure 9: **(a)** that the graph of $\mu_{\hat{m}_k,k}$ for the Henry Hub Natural gas data set moves around the mean value 4.5385 of the real data set. Also, from Figure 9: **(b)**, the values of $\mu_{\hat{m}_k,k}$ for the crude oil data moves around the mean value 54.0093 of the crude oil data set. Likewise, from Figure 9: **(c)**, the values of $\mu_{\hat{m}_k,k}$ for the coal data moves around the mean value 27.1441 of the coal data set. Finally, from Figure 9: **(d)**, the values of $\mu_{\hat{m}_k,k}$ for the ethanol data moves around the mean value 2.1391 of the ethanol data set. This analysis shows that the parameter $\mu_{\hat{m}_k,k}$ is close to the respective mean of the data set at time t_k.

We remark that $\{\mu_{\hat{m}_i,i}\}_{i=0}^{N}$ and $\{a_{\hat{m}_i,i}\}_{i=0}^{N}$ are discrete-time $\epsilon-$ sub-optimal simulated random samples generated by the scheme described at the beginning of Section 7.6.

Next, we show the graph of $s^2_{\hat{m}_k,k}$ for each of the data set: Natural gas, Crude oil, Coal and Ethanol in Figures 10 (a), (b), (c) and (d), respectively.

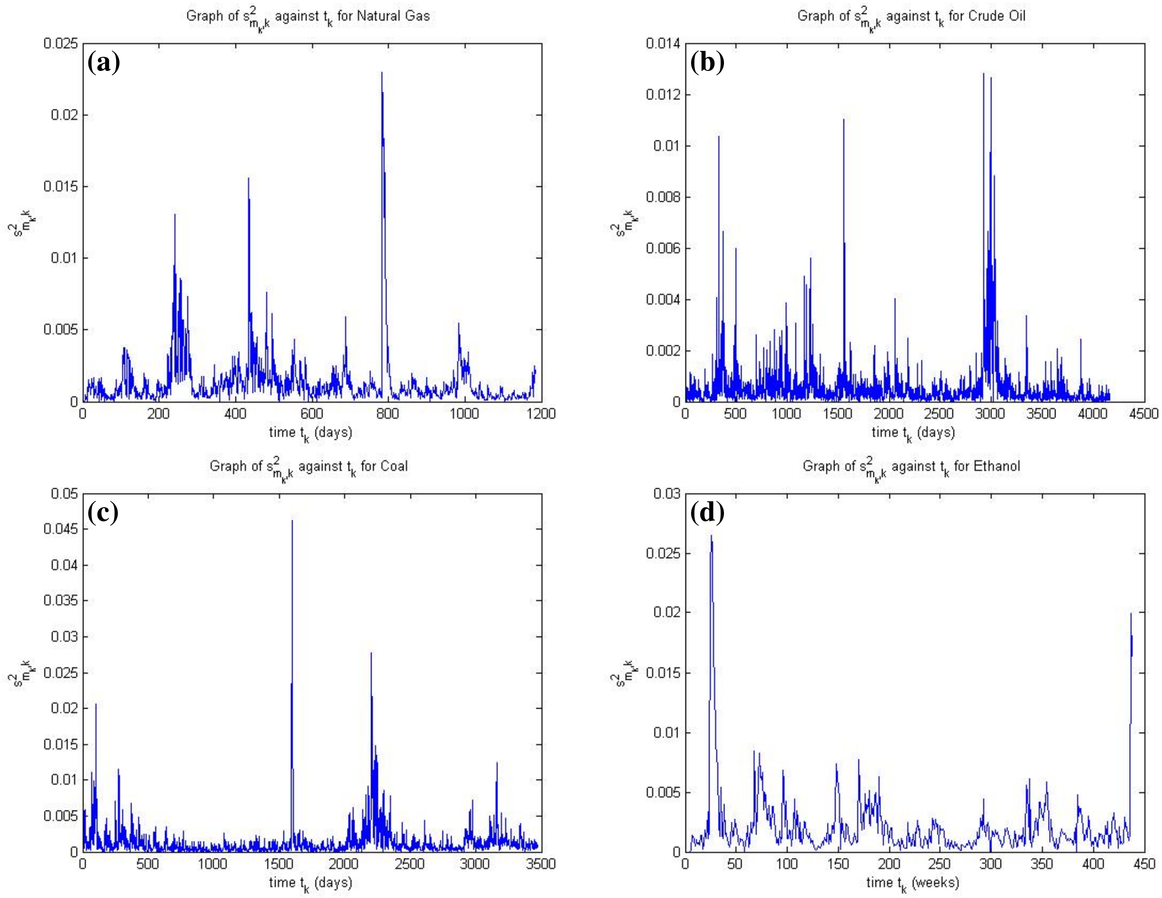


Figure 10.: Moving Variance $s^2_{\hat{m}_k,k}$ against k for three commodities

Figures 10: **(a), (b), (c) and (d)** are graphs of $s^2_{\hat{m}_k,k}$ against time t_k with initial delay $r = 5$ for the daily Henry Hub natural gas data set [24] , daily crude oil data set [23], daily coal data set [22], and weekly ethanol data set [136] respectively. We found these estimates using the discrete time dynamic model (6.8) with $p = 2$, with the usage of $p = 2$ because of the autocorrelation and partial autocorrelation function of the series x, as described in [80]. Using the third part of (6.53), the volatility square at time t_k can be calculated.

The overall descriptive statistics of data sets regarding the energy commodity prices and estimated parameters are recorded in the Table 7.

Table 7: Descriptive Statistics for a, μ and σ^2.

Data Set Y	$\bar{Y}$	Std(Y)	$\overline{\Delta ln(Y)}$	var($\Delta ln(Y)$)	$\bar{a}$	Std(a)	$\bar{\mu}$	Std(μ)	$\overline{\sigma^2}$	std(σ^2)	95% C. I. $\bar{\mu}$
Nat. Gas	4.5504	1.5090	0.0008	0.0015	0.1867	0.3013	4.5538	2.3565	0.0013	0.0017	(4.4196, 4.6880)
Crude Oil	54.0093	31.0248	0.0003	0.0006	0.0215	0.0517	54.0307	37.4455	0.0005	0.0008	(51.8978, 56.1636)
Coal	27.1441	17.8394	0.0003	0.0015	0.0464	0.0879	27.0567	21.3506	0.0014	0.0022	(25.8405, 28.2729)
Ethanol	2.1391	0.4455	0.0011	0.0020	0.3167	0.8745	2.1666	0.7972	0.0018	0.0030	(2.0919,2.2414)

Table 7 shows the descriptive statistics for a, μ and σ^2 with time delay $r = 5$. Note that the mean value of the estimated samples $\{a_{\hat{m}_i,i}\}_{i=0}^{N}$, $\{\mu_{\hat{m}_i,i}\}_{i=0}^{N}$ and $\{\sigma^2_{\hat{m}_i,i}\}_{i=0}^{N}$ are $\bar{a} = \frac{1}{N}\sum_{i=0}^{N} a_{\hat{m}_i,i}$, $\bar{\mu} = \frac{1}{N}\sum_{i=0}^{N} \mu_{\hat{m}_i,i}$ and $\overline{\sigma^2} = \frac{1}{N}\sum_{i=0}^{N} \sigma^2_{\hat{m}_i,i}$, respectively. $\bar{a}$, $\bar{\mu}$, and $\overline{\sigma^2}$ are referred to as aggregated parameter estimates of a, μ, and σ^2 over the given entire finite interval of time, respectively. $\bar{\mu}$ is the descriptive statistics of the parameter μ estimated in column 8, while $\overline{\sigma^2}$ is the descriptive statistics of the parameter σ^2 estimated in column 10. Moreover, $\bar{\mu}$ is approximately close to the overall descriptive statistics of the mean $\bar{Y}$ of the real data set for each of the energy commodities shown in column 2. Also, $\overline{\sigma^2}$ is approximately close to the overall descriptive statistics of the variance of $\Delta \ln(Y) = \ln(Y_i) - \ln(Y_{i-1})$ in Column 5. Moreover, column 12 shows that the mean of the actual data set in Column 2 falls within the 95% confidence interval of $\bar{\mu}$. This exhibits that the parameter $\mu_{\hat{m}_k,k}$ is the mean level of y_k at time t_k.

We have used the the estimated parameters $a_{\hat{m}_k,k}$, $\mu_{\hat{m}_k,k}$, and $\sigma^2_{\hat{m}_k,k}$, in Figures 8, 9, and 10, respectively to simulate the daily natural gas data set, daily crude oil data set, daily coal data set, and weekly ethanol data set.

In fact, developing the code and flowchart described in $C.4$ and the parameters described in Figures 8, 9 and 10, we simulate the daily Natural Gas data set, daily Crude Oil data set, the daily Coal data set and weekly ethanol data set.

For this purpose, we pick $\epsilon = 0.01$; for each time t_k, the estimates of the simulated prices $y^s_{\hat{m}_k,k}$ are computed by determining the sub-optimal admissible set of m_k-size local conditional sample $\mathcal{M}_k$ defined in (7.7). Among these collected values, the value that gives the minimum Ξ_{m_k,k,y_k} is recorded as $\hat{m}_k$. If condition (7.7) is not met at time t_k, the value of m_k where the minimum $\min_{m_k} \Xi_{m_k,k,y_k}$ is attained, is recorded as $\hat{m}_k$. The $\epsilon-$ level sub-optimal estimates of the parameters $\hat{a}_{m_k,k}$, $\hat{\mu}_{m_k,k}$ and $\hat{\sigma}^2_{m_k,k}$ at $\hat{m}_k$ are also recorded as $a_{\hat{m}_k,k}$, $\mu_{\hat{m}_k,k}$ and $\sigma^2_{\hat{m}_k,k}$, the value of $y^s_{m_k,k}$ at time t_k and $\hat{m}_k$ corresponding to $a_{\hat{m}_k,k}$, $\mu_{\hat{m}_k,k}$ and $\sigma^2_{\hat{m}_k,k}$ is also recorded as the $\epsilon-$ sub-optimal simulated value $y^s_{\hat{m}_k,k}$ as an estimate of y_k. A detailed algorithm is given in Appendix $C.4$.

Finally, in Table 8, we show the results of the real, simulated prices using the local lagged adapted generalized method of moment (LLGMM) and the simulated price using the aggregated parameter estimates $\bar{a}$, $\bar{\mu}$, and $\overline{\sigma^2}$ in Table 7, Column 6, 8, and 10, respectively for the energy commodity price. This estimate is derived using the discretized model

$$y_i^{ag} = y_{i-1}^{ag} + \bar{a}(\bar{\mu} - y_{i-1}^{ag})y_{i-1}^{ag}\Delta t + \overline{\sigma^2}^{1/2} y_{i-1}^{ag}\Delta W_i \tag{7.8}$$

For the rest of this study, we define this estimate y_k^{ag} at time t_k by the aggregated GMM simulated estimates (AGMM).

Table 8: Real, Simulation using LLGMM prices, and Simulation using AGMM.

t_k	Natural gas			t_k	Crude oil			t_k	Coal			t_k	Ethanol		
	Real	Simulated $y^s_{\hat{m}_k,k}$ (LLGMM)	Simulated y_k^{ag} (AGMM)		Real	Simulated $y^s_{\hat{m}_k,k}$ (LLGMM)	Simulated y_k^{ag} (AGMM)		Real	Simulated $y^s_{\hat{m}_k,k}$ (LLGMM)	Simulated y_k^{ag} (AGMM)		Real	Simulated $y^s_{\hat{m}_k,k}$ (LLGMM)	Simulated y_k^{ag} (AGMM)
5	2.216	2.216	2.216	5	25.200	25.200	25.000	5	10.560	10.560	10.000	5	1.190	1.190	1.000
6	2.260	2.253	2.276	6	25.100	25.077	25.501	6	10.240	10.436	10.150	6	1.150	1.174	1.233
7	2.244	2.241	2.255	7	25.950	25.606	25.612	7	10.180	10.325	10.555	7	1.180	1.180	1.321
8	2.252	2.249	2.255	8	25.450	25.494	26.011	8	9.560	10.072	10.160	8	1.160	1.148	1.277
9	2.322	2.329	2.291	9	25.400	25.411	26.038	9	8.750	8.338	10.610	9	1.190	1.196	1.318
10	2.383	2.376	2.362	10	25.100	24.981	25.099	10	9.060	9.072	10.936	10	1.190	1.209	1.395
11	2.417	2.417	2.201	11	24.800	24.763	25.715	11	8.880	9.084	10.624	11	1.225	1.186	1.398
12	2.559	2.534	2.182	12	24.400	24.301	25.670	12	9.440	9.581	10.174	12	1.220	1.217	1.473
13	2.485	2.554	2.022	13	23.850	24.862	26.176	13	10.310	9.739	9.807	13	1.290	1.250	1.489
14	2.528	2.525	2.016	14	23.850	23.961	26.142	14	9.810	9.633	9.548	14	1.410	1.320	1.459
15	2.616	2.615	2.057	15	23.850	24.010	26.602	15	9.060	9.197	9.904	15	1.470	1.392	1.451
16	2.523	2.478	2.122	16	23.900	24.071	26.094	16	8.750	8.806	9.888	16	1.530	1.461	1.378
17	2.610	2.638	2.181	17	24.500	24.554	26.051	17	8.820	8.879	9.878	17	1.630	1.545	1.275
18	2.610	2.606	2.265	18	24.800	24.795	25.973	18	9.560	9.326	10.095	18	1.750	1.7433	1.284
19	2.610	2.614	2.356	19	24.150	24.165	26.385	19	8.820	8.749	10.158	19	1.750	1.858	1.189
20	2.699	2.726	2.430	20	24.200	23.971	25.817	20	8.820	8.774	10.180	20	1.840	1.886	1.224
...	...	....	...	...	...	...	....	...	...	...	...	...	...	...	...
...	....	....	...	....	...	....	....	...	...	...	...	...	...	...	...
1145	5.712	5.709	5.356	2440	57.350	57.298	64.878	2865	29.310	29.065	28.288	375	2.073	2.019	2.068
1146	5.588	5.592	5.464	2441	56.740	56.650	64.333	2866	28.680	28.619	26.839	376	2.020	2.003	1.948
1147	5.693	5.650	5.610	2442	57.550	57.613	64.350	2867	26.770	28.408	26.448	377	2.073	2.094	1.868
1148	5.791	5.786	5.489	2443	59.090	59.152	64.319	2868	27.450	27.480	26.555	378	2.065	2.076	1.898
1149	5.614	5.458	5.682	2444	60.270	58.926	65.331	2869	27.000	27.250	27.808	379	2.055	2.061	1.803
1150	5.442	5.460	6.047	2445	60.750	59.675	64.43	2870	26.670	26.544	26.804	380	2.209	2.169	1.869
1151	5.533	5.571	6.192	2446	58.410	59.408	66.356	2871	26.510	26.497	27.429	381	2.440	2.208	1.764
1152	5.378	5.397	6.251	2447	58.720	58.917	65.789	2872	26.480	26.463	28.481	382	2.517	2.220	1.687
1153	5.373	5.374	6.085	2448	58.640	58.502	61.865	2873	25.150	25.781	29.022	383	2.718	2.362	1.744
1154	5.382	5.420	5.901	2449	57.870	58.721	64.171	2874	25.570	25.615	28.829	384	2.541	2.687	1.716
1155	5.507	5.501	5.986	2450	59.130	58.985	64.001	2875	25.880	25.948	29.549	385	2.566	2.607	1.785
1156	5.552	5.551	5.632	2451	60.110	60.087	64.234	2876	25.240	25.451	29.080	386	2.626	2.549	1.716
1157	5.310	5.272	5.525	2452	58.940	58.858	64.419	2877	25.000	24.649	29.392	387	2.587	2.606	1.816
1158	5.338	5.348	5.183	2453	59.930	59.390	62.080	2878	25.080	24.984	28.834	388	2.628	2.624	1.761
1159	5.298	5.353	5.024	2454	61.180	60.283	59.690	2879	25.050	25.158	29.122	389	2.587	2.556	1.909
1160	5.189	5.207	5.025	2455	59.660	59.939	60.680	2880	25.890	25.835	31.099	390	2.536	2.546	1.969

Next, we show the graph of the simulated data set using the LLGMM method for each of the commodities in Figure 11.

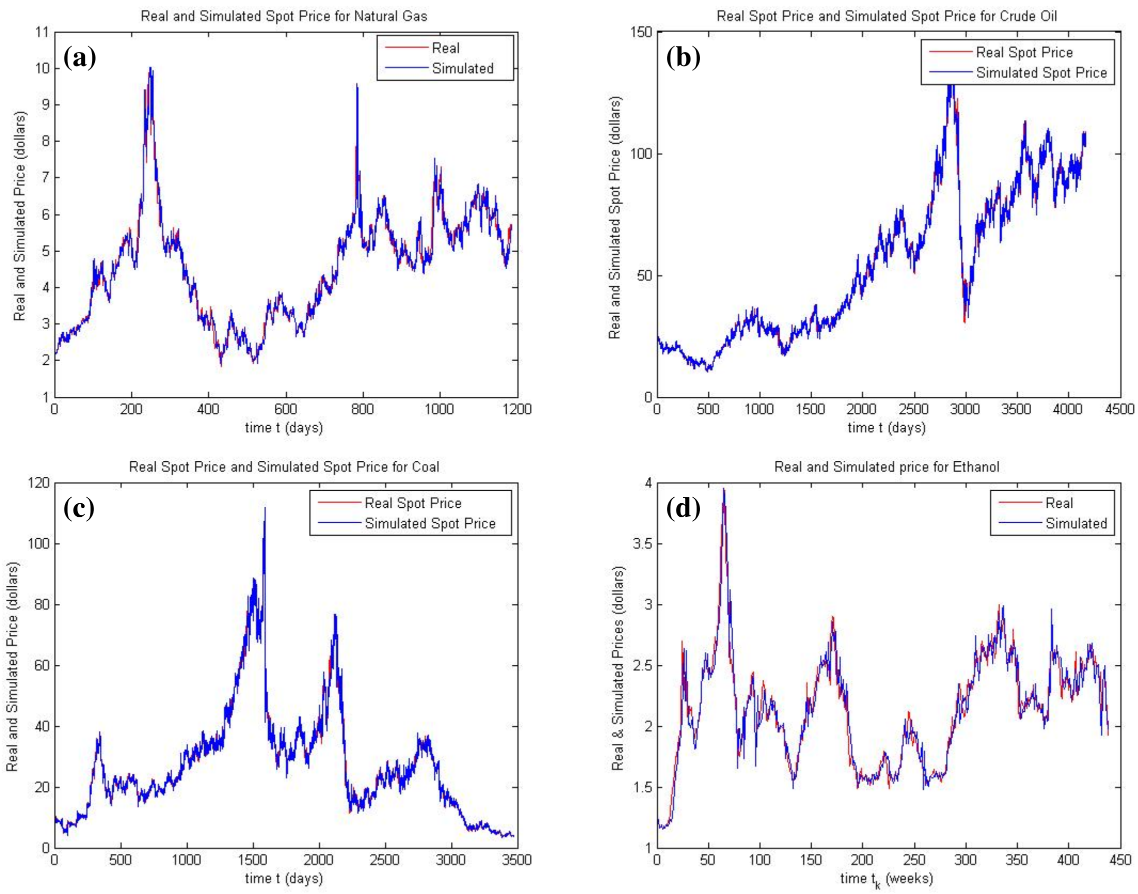


Figure 11.: Real and Simulated Prices: $r = 5$.

Figures 11: **(a), (b), (c) and (d)** show the graph of the Real and Simulated Spot Prices for the daily Henry Hub natural gas data set [24] , daily crude oil data set [23], daily coal data set [22], and weekly ethanol data set [136] respectively. The red line represents the real data set y_k, while the blue line represent the simulated data set $y^s_{\hat{m}_k,k}$. The root mean square error of the simulation for the Henry Hub Natural gas data set, the Crude Oil data set, the Coal and Ethanol data set are given by 0.021, 0.013, 0.015, and 0.046 respectively. Here, we begin by using a starting delay of $r = 5$. The simulation starts from $t_r = t_5$. It is clear that the graph fits well, but there are still some regions where the simulation does not capture the real data well. Therefore, this gap is analyzed by increasing the magnitude of time delay.

The following graphs show the Real and Simulated Spot Prices for the daily Henry Hub natural gas data set [24] , daily crude oil data set [23], daily coal data set [22], and weekly ethanol data set [136], respectively, for the case where $r = 10$.

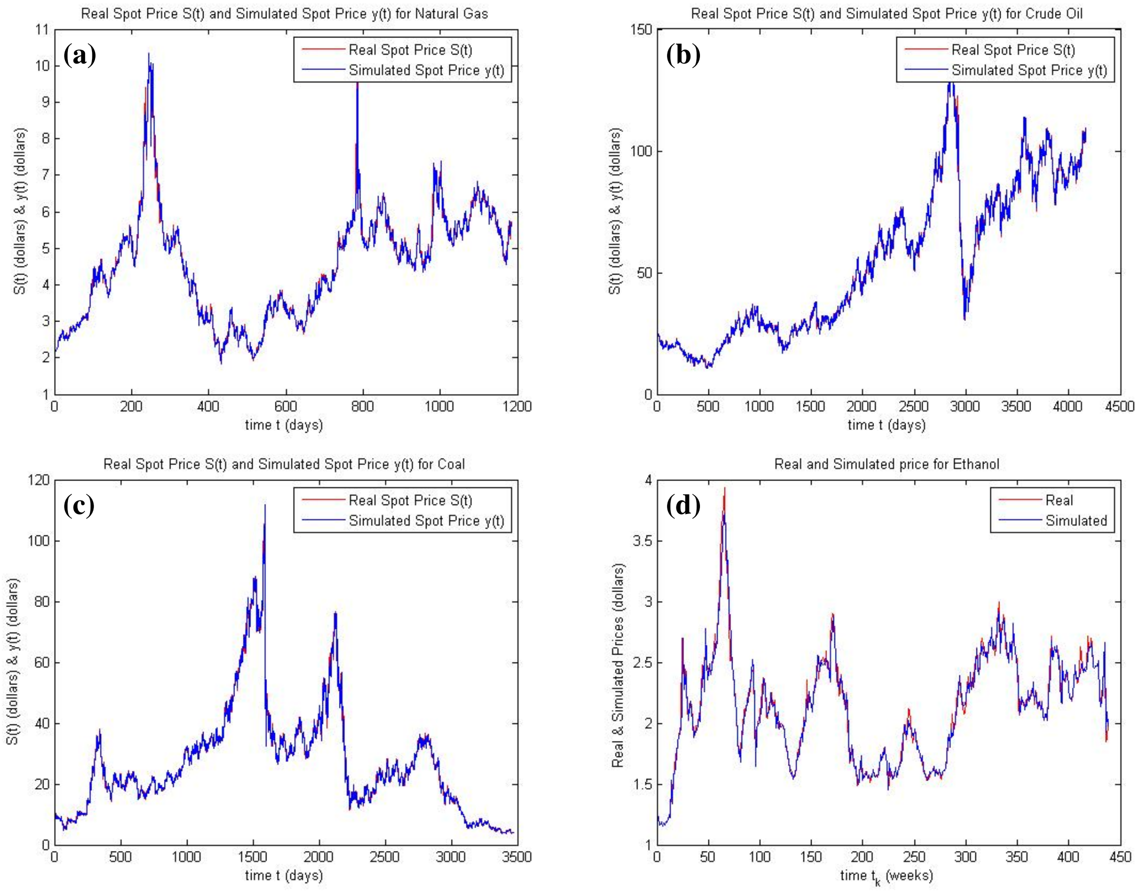


Figure 12.: Real and Simulated Prices: $r = 10$.

Figures 12: **(a), (b), (c) and (d)** show the graph of the Real and Simulated Spot Prices for the daily Henry Hub Natural gas data set [24] , daily Crude Oil data set [23], daily Coal data set [22], and weekly ethanol data set [136], respectively, for $r = 10$. The red line represents the real data set y_k while the blue line represent the simulated data set $y^s_{\hat{m}_k,k}$. The root mean square error of the simulation for the Henry Hub Natural gas data set, the Crude Oil data set, the Coal data set, and ethanol data set are given by 0.004, 0.001, 0.002 and 0.006, respectively.

REMARK 21 Several other delays were tested and it was found that as the delay r increases, the root mean square error decreases, significantly. Moreover, the curve fitting appears to be better. For example, for starting delay of 20, the root mean square error of the simulation for the Henry Hub natural gas data set, the crude oil data set, coal data set and ethanol data set are given by 2×10^{-4}, 10^{-5}, 10^{-4}, and 5×10^{-4}, respectively. Furthermore, the simulation results show that the price of a commodity is affected by its volatility $\sigma^2_{\hat{m}_k,k}$, the rate and mean level parameters $a_{\hat{m}_k,k}$ and $\mu_{\hat{m}_k,k}$, respectively.

In Figure 13, we show a comparison between the real data set, simulated price using the local lagged adaptive generalized method (LLGMM) and the simulated price (AGMM) using the aggregated parameter estimates $\bar{a}$, $\bar{\mu}$, and $\overline{\sigma^2}$ in Table 7, Column 6, 8, and 10, respectively.

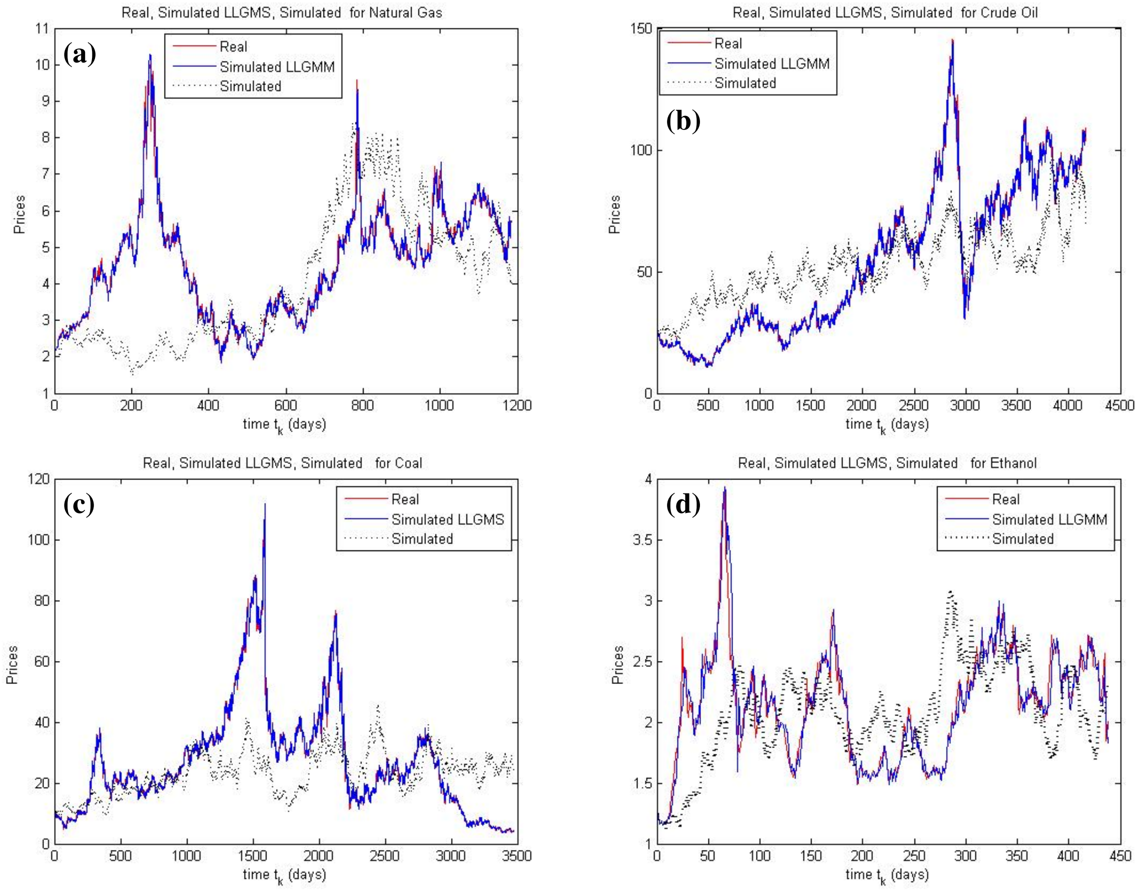


Figure 13.: Real, Simulated Prices using (LLGMM), and Simulated Prices using AGMM.

Figures 13: **(a), (b), (c) and (d)** show the graph of the Real, simulated prices using the local lagged adaptive generalized method (LLGMM), and the simulated price using the average of the parameters for Henry Hub Natural gas data set [24] , daily Crude Oil data set [23], daily Coal data set [22], and weekly ethanol data set [136], respectively, for $r = 5$. The red line represents the real data set y_k, the blue line represent the simulated prices using LLGMM method, while the black line represent the simulated price (AGMM) using the aggregated parameter estimates $\bar{a}$, $\bar{\mu}$, and $\overline{\sigma^2}$ in Table 7, Column 6, 8, and 10, respectively. From these simulated graphs, it is clear that the LLGMM simulation results are more realistic than the AGMM simulation results. This exhibits the power of LLGMM over the AGMM.

REMARK 22 A code similar to the flowchart described in $C.4$ is designed to exhibit the flowchart algorithm. All the codes for the parameter estimation, simulations and forecasting are written and tested using Matlab program. Due to the online control nature of m_k in our model, it is worth mentioning that the running times for each of the four commodities: Natural gas, Crude oil, Coal and Ethanol depend on the robustness of the data. The average running time for each data set is 25 minutes.

In reference to Remark 16, we compare the estimates $s^2_{\hat{m}_k,k}$ with the estimate derived from the usage of a GARCH(1,1) model described in [9] which is defined by

$$\begin{aligned} z_t|\mathcal{F}_{t-1} &\sim \mathcal{N}(0,h_t), \\ h_t &= \alpha_0+\alpha_1 h_{t-1}+\beta_1 z_{t-1}^2,\ \ \alpha_0>0,\ \alpha_1,\beta_1\geq 0. \end{aligned} \tag{7.9}$$

The parameters α_0, α_1, and β_1 of the GARCH(1,1) conditional variance model (7.9) for x for the four commodities natural gas, crude oil, coal, and ethanol are estimated. The estimates of the parameters are given in Table 9.

Table 9: Parameter estimates for GARCH(1,1) Model (7.9).

Data Set	α_0	α_1	β_1
Natural Gas	6.863×10^{-5}	0.853	0.112
Crude Oil	9.622×10^{-5}	0.917	0.069
Coal	3.023×10^{-5}	0.903	0.081
Ethanol	4.152×10^{-4}	0.815	0.019

Table 9 shows the parameter estimates for GARCH(1,1) Model

We later show a side by side comparison of $s^2_{\hat{m}_k,k}$ and the volatility described by GARCH(1,1) model described in (7.9) with coefficients in Table 9.

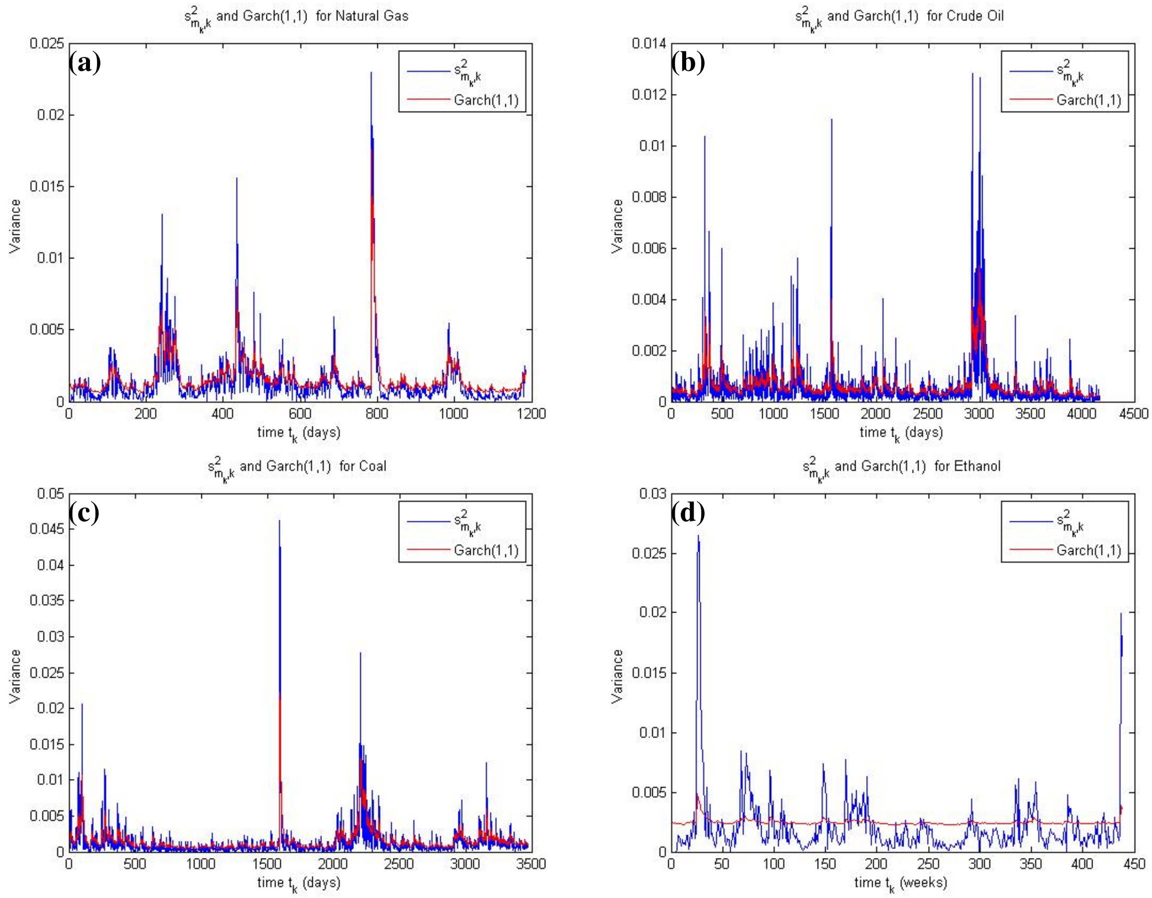


Figure 14.: $s^2_{\hat{m}_k,k}$ and GARCH(1,1) .

Figures 14: **(a), (b) and (c)** are graphs of $s^2_{\hat{m}_k,k}$ and GARCH(1,1) model against time t_k for the daily Henry Hub natural gas data set [24] , daily crude oil data set [23], daily coal data set [22] and weekly ethanol data set [136] respectively. The blue line shows the graph of estimates for $s^2_{\hat{m}_k,k}$ and the red line shows the graph of estimates for GARCH(1,1) model. The GARCH model does not clearly estimate volatility as heavily evidenced in Figure 14 **(d)**. In fact, it demostrated insensitivity. The presented analysis suggests that the GARCH model is ineffective in comparison with the framework of moving average process.

We also compare the simulations in Figure 11 with the simulations using the GARCH(1,1) model (7.9) as the conditional variance. The following figure exhibits the comparison.

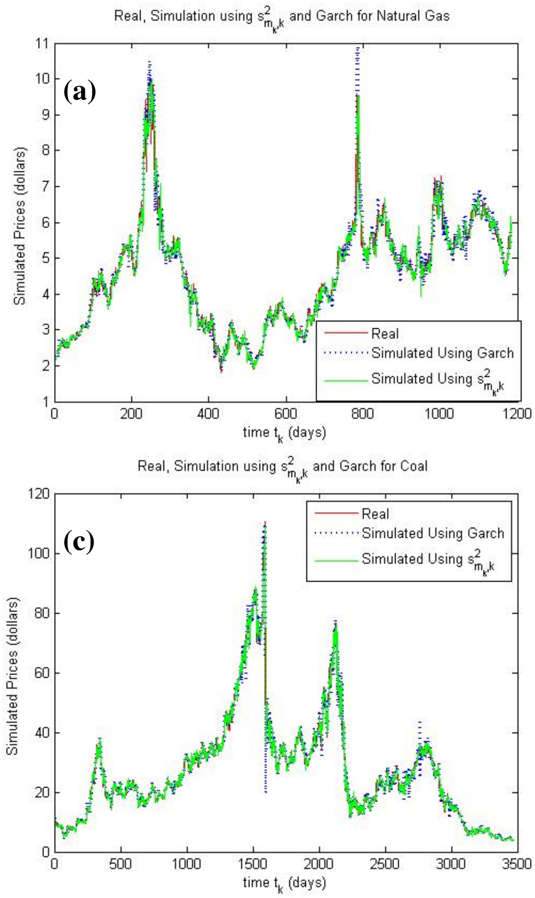


Figure 15.: Simulation derived by using $s^2_{\hat{m}_k,k}$ and GARCH(1,1)

Figures 15: **(a), (b), (c) and (d)** are graphs of the simulations using $s^2_{\hat{m}_k,k}$ and GARCH(1,1) model to estimate the volatility process for the daily Henry Hub Natural gas data set [24] , daily Crude oil data set [23], daily Coal data set [22], and weekly Ethanol data set [136], respectively. The blue line shows the graph of estimates for the simulations using GARCH(1,1) model to simulate the volatility, the green line is our simulated estimates described in Figure 11, and the red line shows the real data set. It can be seen that the GARCH model fails to capture most of the spikes in the data set. Moreover, the GARCH model creates significant errors by over-and-under estimating the variance. These spikes were perfectly captured in Figure 11 where we use the discrete-time dynamic model of local sample variance statistics process to estimate the volatility process. This illustrates that the dynamic statistic model works better than the GARCH volatility model.

7.7 Applications: U. S. Treasury Bill Yield Interest Rate and U. S. Eurocurrency Exchange Rate Data Sets

In this subsection, we apply the conceptual computational algorithm discussed above to estimate the parameters in (6.42) using the real time Treasury bill yield data sets [128] and the US dollar Eurocurrency data set [129] collected from Forex database.

Table 10: Estimates $\hat{m}_k$, $\beta_{\hat{m}_k,k}$, $\mu_{\hat{m}_k,k}$, $\delta_{\hat{m}_k,k}$, $\sigma_{\hat{m}_k,k}$, $\gamma_{\hat{m}_k,k}$ for U. S. Treasury Bill Interest Rate.

t_k	Interest Rate					
	$\hat{m}_k$	$\beta_{\hat{m}_k,k}$	$\mu_{\hat{m}_k,k}$	$\delta_{\hat{m}_k,k}$	$\sigma_{\hat{m}_k,k}$	$\gamma_{\hat{m}_k,k}$
5	2	1.342	6.1344	1.23	0.0650	1.46
6	2	-1.4536	7.7969	1.4600	0.0740	1.4963
7	3	-1.9051	15.5056	1.4600	0.0973	1.4942
8	5	0.3703	-1.8563	1.4600	0.2681	0.5218
9	5	0.3209	-1.46591	1.4600	0.2674	1.1845
10	5	0.8104	-4.0798	1.4600	0.2605	1.0916
11	4	2.0471	-10.3264	1.4600	0.0948	1.4632
12	4	2.2642	-11.4387	1.4600	0.0692	1.4335
13	3	-3.5423	17.8429	1.4600	0.0660	1.4989
14	5	0.7275	-3.7090	1.4600	0.0734	1.4835
15	5	0.9500	-4.8475	...	0.0688	1.4968
16	5	1.8567	-9.4523	1.46000	0.0836	1.4998
17	5	-0.0187	0.1323	1.46000	0.1289	1.4990
18	5	-0.2657	1.4233	1.4600	0.1148	1.4168
19	4	-1.6380	-8.3661	1.4600	0.2017	1.0316
20	4	-0.3631	-1.9923	1.4600	0.2236	0.9800
....	....	....	.....	....	....	.....
....	....	....	.....	....	....	.....
420	5	3.8416	-18.3309	1.4600	0.1187	1.4986
421	4	2.0695	-9.8104	1.4600	0.1594	1.4906
422	4	1.4882	-7.0240	1.4600	0.1643	1.2216
423	4	1.1992	-5.5523	1.4600	0.1899	1.3399
424	4	1.2800	-5.9315	1.4600	0.1814	1.4736
425	5	0.4408	-1.9812	1.4600	0.1901	1.4961
426	5	-0.5137	2.4739	1.4600	0.1616	1.4972
427	5	0.0508	-0.1078	1.4600	0.1547	1.4994
428	5	-0.0623	0.3913	1.4600	0.1174	1.4975
429	5	0.1269	-0.4112	1.4600	0.1175	1.5050
430	5	0.1828	-0.6611	1.4600	0.0856	1.4983
431	5	0.4780	-1.9674	1.4600	0.0765	0.1008
432	5	0.5655	-2.3219	1.4600	0.0719	1.0406
433	5	1.4002	-5.8583	1.4600	0.0839	1.3915
434	5	1.9290	-8.0357	1.4600	0.1145	1.4369
435	4	0.6095	-2.4585	1.4600	0.2260	1.2307

Table 11: Estimates $\hat{m}_k$, $\beta_{\hat{m}_k,k}$, $\mu_{\hat{m}_k,k}$, $\delta_{\hat{m}_k,k}$, $\sigma_{\hat{m}_k,k}$, $\gamma_{\hat{m}_k,k}$ for U.S. Eurocurrency Exchange Rate.

t_k	US Eurocurrency Exchange Rate					
	$\hat{m}_k$	$\beta_{\hat{m}_k,k}$	$\mu_{\hat{m}_k,k}$	$\delta_{\hat{m}_k,k}$	$\sigma_{\hat{m}_k,k}$	$\gamma_{\hat{m}_k,k}$
5	1	0	0	1.4892	0	1.2404
6	2	-0.8614	0.6922	1.4892	0.0203	1.4636
7	2	-1.5672	1.2363	1.4892	0.0206	1.4996
8	2	1.9150	-1.4001	1.4892	0.0614	-0.5243
9	4	0.8849	-0.6401	1.4892	0.0562	0.3397
10	2	-9.3647	6.8531	1.4892	0.0170	1.4967
11	5	1.2713	-0.9156	1.4892	0.0422	1.4892
12	4	1.6414	-1.1824	1.4892	0.0138	0.9651
13	4	2.9985	-2.1536	1.4892	0.0153	-1.7912
14	4	3.2093	-2.3045	1.4892	0.0091	0.7281
15	4	0.3170	-0.2468	1.4892	0.0411	1.4999
16	3	0.6723	-0.5221	1.4892	0.0482	1.3425
17	5	0.4002	-0.3101	1.4892	0.0401	0.2017
18	5	0.4562	-0.3666	1.4892	0.0425	0.3349
19	5	1.3955	-1.0776	1.4892	0.0467	1.4892
20	4	1.9070	-1.4757	1.4892	0.0166	1.4933
....	....	....	.....	....	....	.....
....	....	....	.....	....	....	.....
155	3	2.3290	-1.8636	1.4892	0.0049	1.0646
156	4	2.8385	-2.2656	1.4892	0.0132	1.0755
157	5	-0.4474	0.3668	1.4892	0.0280	1.5024
158	5	1.5170	-1.2021	1.4892	0.0356	1.5540
159	5	1.6898	-1.3382	1.4892	0.0356	1.1792
160	5	2.2439	-1.7720	1.4892	0.0357	1.4624
161	4	-0.5436	0.4407	1.4892	0.0185	1.4976
162	3	0.3131	-0.2332	1.4892	0.0144	1.4375
163	3	2.9744	-2.3248	1.4892	0.0240	1.2378
164	5	1.4940	-1.1744	1.4892	0.0182	0.7835
165	5	2.7775	-2.1773	1.4892	0.0225	1.4995
166	5	1.9622	-1.5328	1.4892	0.0255	1.1941
167	4	0.8564	-0.6573	1.4892	0.0342	0.7930
168	4	0.7604	-0.5650	1.4892	0.0353	1.4061
169	5	0.5422	-0.3945	1.4892	0.0329	0.6155
170	4	0.1764	-0.1124	1.4892	0.0257	1.3552

Tables 10-11 show the estimates for the ϵ- sub-optimal size $\hat{m}_k$, the parameters $\beta_{\hat{m}_k,k}$, $\mu_{\hat{m}_k,k}$, $\delta_{\hat{m}_k,k}$, $\sigma_{\hat{m}_k,k}$, and $\gamma_{\hat{m}_k,k}$ for the U. S. Treasury Bill Yield Interest Rate and U. S. Eurocurrency Exchange Rate, respectively. The initial real data time is $t_r = t_5$.

Using $\epsilon = 1 \times 10^{-3}$, $r = 5$, and $p = 2$, the $\epsilon-$ level sub-optimal estimates of parameters β, μ, δ, σ and γ for each Treasury bill real data set and U.S. Eurocurrency rate data sets are exhibited in Tables 10 and 11, respectively.

Next, we show the graphs of $\beta_{\hat{m_k},k}$, $\mu_{\hat{m_k},k}$, $\delta_{\hat{m_k},k}$, $\sigma_{\hat{m_k},k}$, and $\gamma_{\hat{m_k},k}$ for the Monthly Treasury bill data set and Monthly U. S. Eurocurrency data set.

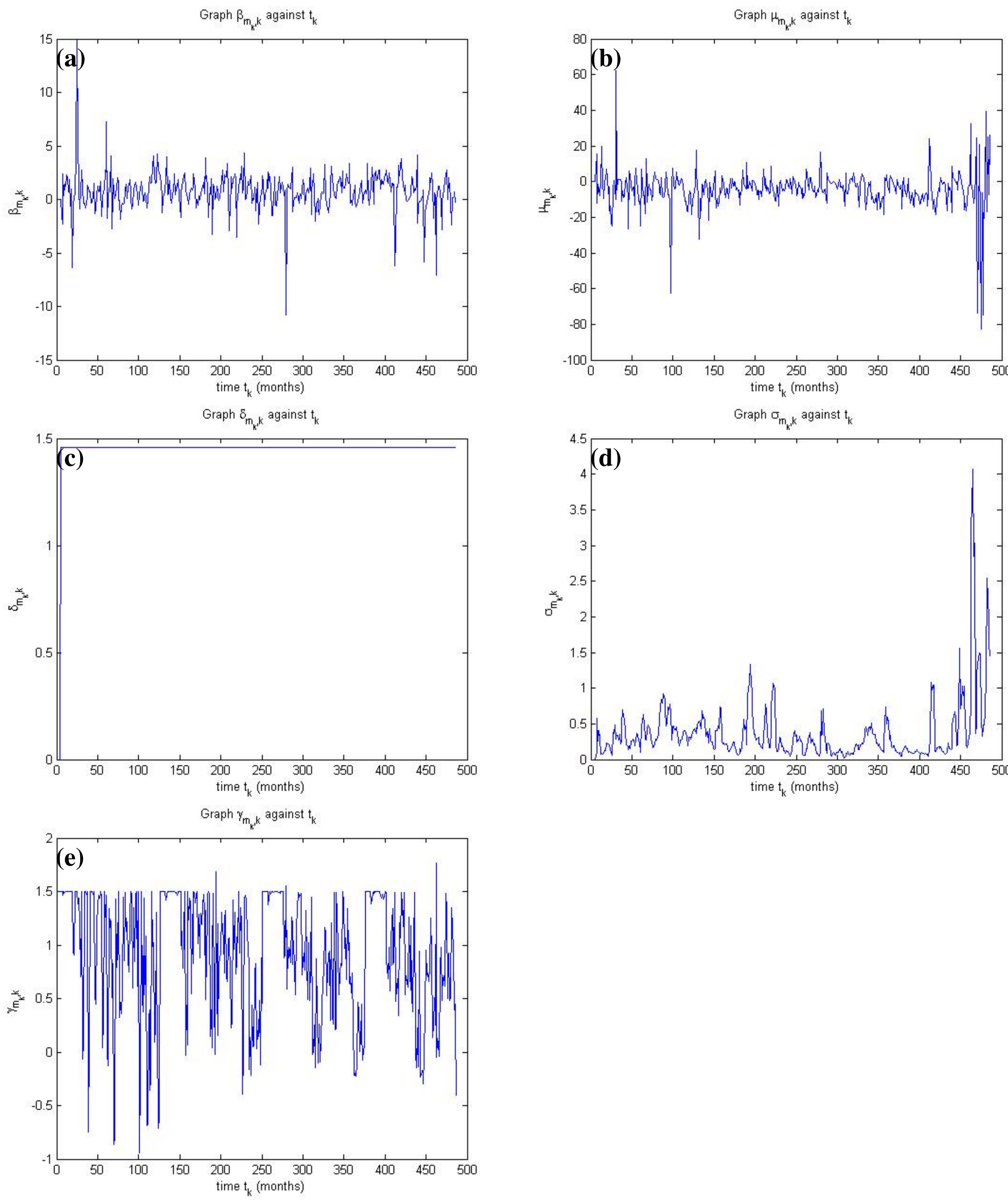


Figure 16.: $\beta_{\hat{m_k},k}$, $\mu_{\hat{m_k},k}$, $\delta_{\hat{m_k},k}$, $\sigma_{\hat{m_k},k}$, and $\gamma_{\hat{m_k},k}$ for Interest rate.

Figures 16: **(a), (b), (c) and (d)** are the graphs of the parameters $\beta_{\hat{m}_k,k}$, $\mu_{\hat{m}_k,k}$, $\delta_{\hat{m}_k,k}$, $\sigma_{\hat{m}_k,k}$, and $\gamma_{\hat{m}_k,k}$ against time t_k for the U.S. Treasury bill yield respectively.

The next figures show the graphs of the parameters $\beta_{\hat{m}_k,k}$, $\mu_{\hat{m}_k,k}$, $\delta_{\hat{m}_k,k}$, $\sigma_{\hat{m}_k,k}$, and $\gamma_{\hat{m}_k,k}$ for the Eurocurrency Exchange rate respectively.

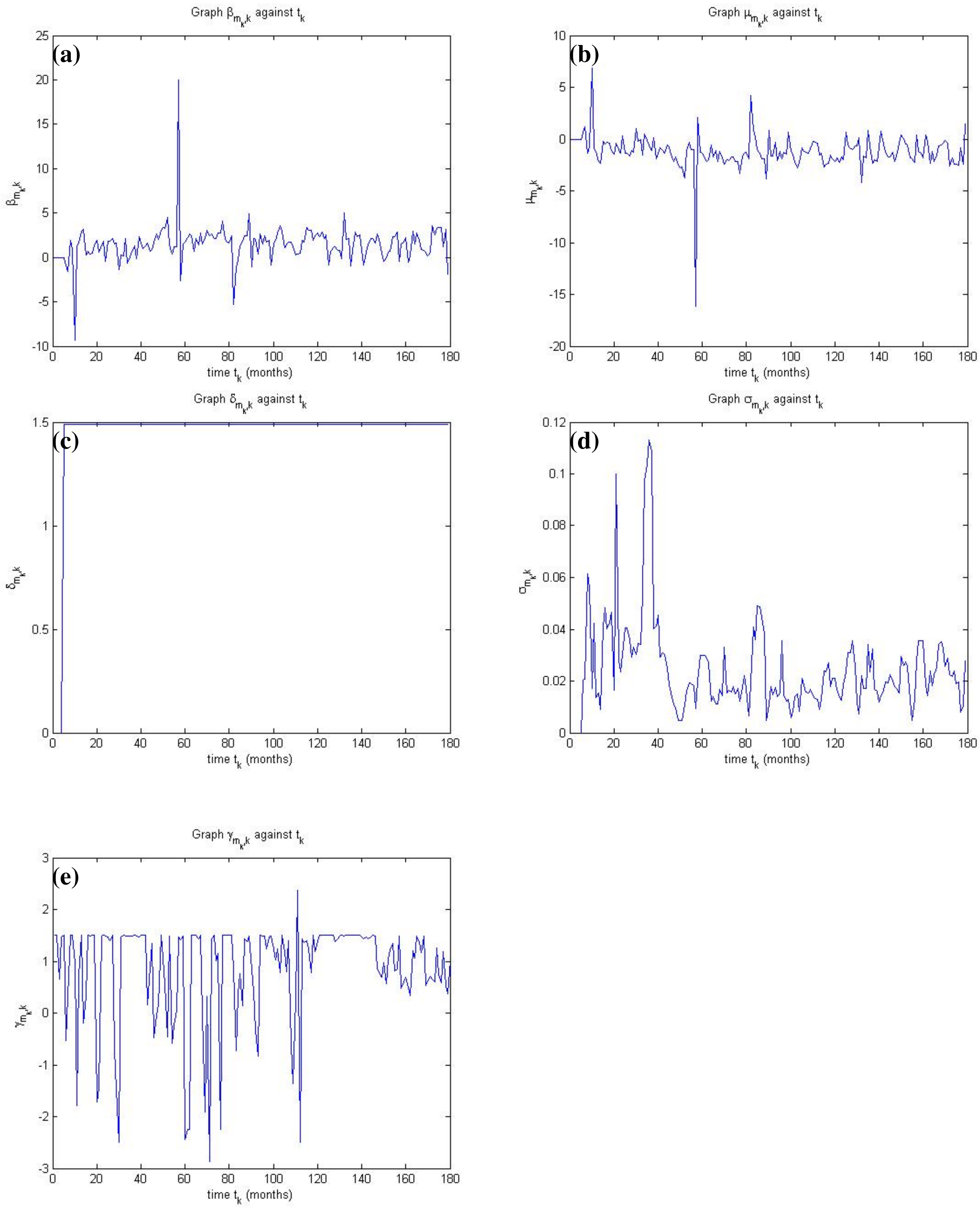


Figure 17.: $\beta_{\hat{m}_k,k}$, $\mu_{\hat{m}_k,k}$, $\delta_{\hat{m}_k,k}$, $\sigma_{\hat{m}_k,k}$, and $\gamma_{\hat{m}_k,k}$ for US Eurocurrency.

Figures 17: **(a), (b), (c) and (d)** are the graphs of the parameters $\beta_{\hat{m}_k,k}$, $\mu_{\hat{m}_k,k}$, $\delta_{\hat{m}_k,k}$, $\sigma_{\hat{m}_k,k}$, and $\gamma_{\hat{m}_k,k}$ against time t_k for the US Eurocurrency exchange rate respectively.

The overall descriptive statistics of data sets regarding U. S. Treasury Bill Yield Interest Rate and U. S. Eurocurrency Exchange Rate are recorded in the following table.

Table 12: Descriptive Statistics for $\beta_{\hat{m}_k,k}$, $\mu_{\hat{m}_k,k}$, $\delta_{\hat{m}_k,k}$, $\sigma_{\hat{m}_k,k}$, and $\gamma_{\hat{m}_k,k}$ for Interest rate data set.

$\bar{Y}$	Std(Y)	$\bar{\beta}$	Std(β)	$\bar{\mu}$	Std(μ)	$\bar{\delta}$	Std(δ)	$\bar{\sigma}$	std(σ)	$\bar{\gamma}$	Std(γ)
0.05667	0.0268	0.8739	1.8129	-3.8555	8.7608	1.4600	0.00	0.3753	0.5197	1.4877	0.1357

Table 13: Descriptive Statistics for $\beta_{\hat{m}_k,k}$, $\mu_{\hat{m}_k,k}$, $\delta_{\hat{m}_k,k}$, $\sigma_{\hat{m}_k,k}$, and $\gamma_{\hat{m}_k,k}$ for US Eurocurrency Exchange Rate data.

$\bar{Y}$	Std(Y)	$\bar{\beta}$	Std(β)	$\bar{\mu}$	Std(μ)	$\bar{\delta}$	Std(δ)	$\bar{\sigma}$	std(σ)	$\bar{\gamma}$	Std(γ)
1.6249	0.1337	1.5120	2.1259	-1.1973	1.6811	1.4892	0.00	0.0243	0.0180	1.08476	1.0050

Tables 12 and 13 show the descriptive statistics for $\beta_{\hat{m}_k,k}$, $\mu_{\hat{m}_k,k}$, $\delta_{\hat{m}_k,k}$, $\sigma_{\hat{m}_k,k}$, and $\gamma_{\hat{m}_k,k}$ with time delay $r = 5$ for the U.S. Treasury Bill Yield interest rate data set and the U. S. Eurocurrency exchange rate data set, respectively.

In Table 14, we show the result for the real, simulated data using the local lagged adapted generalized method of moment (LLMM), and the simulated price (AGMM) using the aggregated parameter estimates $\bar{\beta}$, $\bar{\mu}$, $de\bar{l}ta$, $\overline{\sigma}$ and $\bar{\gamma}$ in Table 12 and 13 for the U. S. Treasury Bill Yield interest rate and U. S. Eurocurrency exchange rate respectively. The simulated price using the aggregated parameter (AGMM) satisfies the discrete model

$$y_i^{ag} = y_{i-1}^{ag} + (\bar{\beta} y_{i-1}^{ag} + \bar{\mu}(y_{i-1}^{ag})^{\bar{\delta}}) + \bar{\sigma}(y_{i-1}^{ag})^{\bar{\gamma}} \Delta W_i. \tag{7.10}$$

Table 14: Estimates for Real, Simulated Price using LLGMM, Simulated Price using AGMM method for U.S. Treasury Bill Yield Interest Rate and U.S. Eurocurrency Exchange Rate respectively.

t_k	Interest Rate Data			t_k	Eurocurrency Rate		
	Real	Simulated LLGMM	Simulated AGMM		Real Real	Simulated LLGMM	Simulated AGMM
5	0.0357	0.0357	0.0400	5	1.6830	1.6830	1.6530
6	0.0364	0.0357	0.0390	6	1.7446	1.6830	1.8876
7	0.0384	0.0365	0.0390	7	1.8592	1.7596	1.9952
8	0.0381	0.0378	0.0388	8	1.8702	1.8735	1.8702
9	0.0393	0.0387	0.0399	9	1.8837	1.8884	1.8837
10	0.0393	0.0406	0.0401	10	1.9609	1.9191	1.9609
11	0.0393	0.0389	0.0503	11	1.9375	1.9768	1.7307
12	0.0389	0.0393	0.0612	12	1.9140	1.8514	1.7245
13	0.0380	0.0386	0.0570	13	1.9682	1.9754	1.7191
14	0.0384	0.0391	0.0301	14	1.9236	1.9405	1.6657
15	0.0384	0.0390	0.0384	15	1.7328	1.8728	1.6403
16	0.0392	0.0385	0.0174	16	1.7241	1.8953	1.5961
17	0.0403	0.0392	0.0223	17	1.7074	1.7509	1.5929
18	0.0409	0.0385	0.0299	18	1.6258	1.6628	1.5638
19	0.0438	0.0399	0.0449	19	1.6732	1.6517	1.6017
20	0.0459	0.0444	0.0419	20	1.6732	1.7021	1.6785
...	...	...	..		...	...	...
..	...	...	...		...	...	...
390	0.0503	0.0503	0.0303	119	1.6011	1.6032	1.6029
391	0.0491	0.0517	0.0371	120	1.6371	1.5963	1.6102
392	0.0503	0.0503	0.0501	121	1.6145	1.6177	1.6058
393	0.0501	0.0505	0.0556	122	1.5865	1.5830	1.6061
394	0.0514	0.0492	0.0488	123	1.5985	1.6186	1.5350
395	0.0516	0.0510	0.0506	124	1.5528	1.5541	1.5936
396	0.0505	0.0484	0.0409	125	1.4948	1.5341	1.5197
397	0.0493	0.0503	0.0530	126	1.5138	1.5244	1.6091
398	0.0505	0.0494	0.0532	127	1.4922	1.4558	1.6737
399	0.0514	0.0496	0.0395	128	1.4644	1.4518	1.6588
400	0.0495	0.0512	0.0471	129	1.4675	1.4777	1.5698
402	0.0514	0.0496	0.0184	131	1.4416	1.4440	1.6404
403	0.0516	0.0513	0.0203	132	1.4960	1.4553	1.7092
404	0.0504	0.0483	0.0252	133	1.4787	1.4867	1.5950
405	0.0509	0.0491	0.0228	134	1.4550	1.4264	1.5864

Next, we show the graphs of the simulated data set for the Treasury bill yield interest rate and US Eurocurrency exchange rate data.

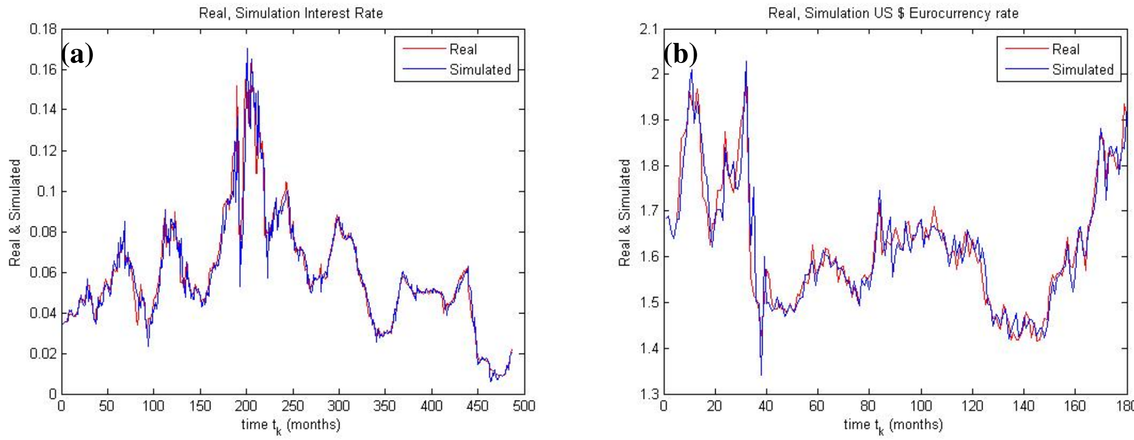


Figure 18.: Real and Simulated price for Interest rate and U.S. Eurocurrency rate.

Figures 18(a) and (b) show the real and simulated price for U.S. Treasury bill yield interest rate and U.S. Eurocurrency exchange rate respectively.

In the work of Chan et al [15], they compared the ex post volatility (defined by the absolute value of the change in Treasury bill yield data set) with the simulated volatility (defined by the square root of the conditional variance implied by the estimates of the the solution of (6.42). It is calculated as $\sigma_{\hat{m}_k,k}\left(y^s_{\hat{m}_k,k}\right)^{\delta_{\hat{m}_k,k}}$). In order to compare our work with Figure 1 of Chan et al [15], we use our approach/scheme to compute the Ex post volatility and simulated volatility for the U.S. Treasury bill yield interest rate data set [128].

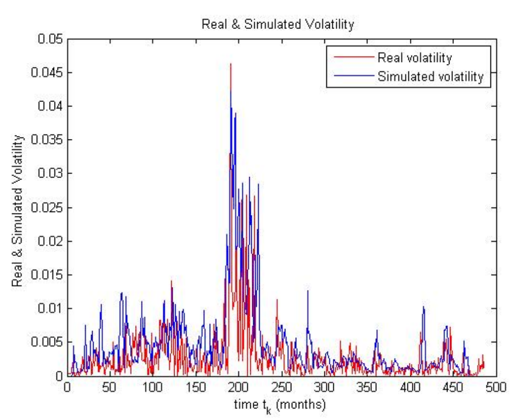


Figure 19.: Ex Post Volatility and Simulated Volatility for Interest Rate.

Figures 19 shows the Ex post volatility and simulated volatility for the U.S. Treasury bill yield interest rate data set [128]. We compare our work with Figure 1 of Chan et al [15]. Their model does not clearly estimate the volatility. It demostrated insensitivity in the sense that it was unable to capture most of the spikes in the interest rate ex post volatility data set.

Finally, in Figure 20, we show the graphs of comparison of the real price, simulated price using the LLGMM method and the simulated price AGMM method.

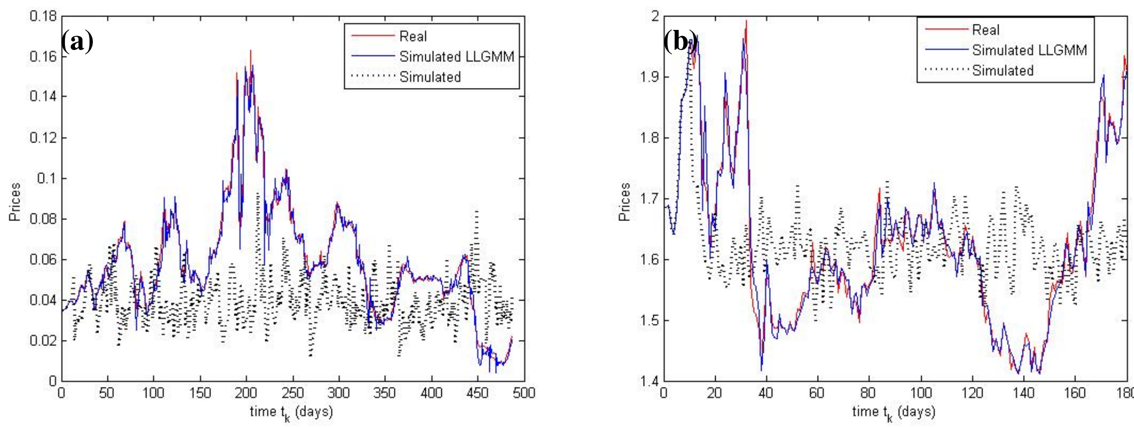


Figure 20.: Real, Simulation using LLGMM, and Simulated Price using AGMM for U.S. Treasury Bill Yield Interest Rate and U.S. Eurocurrency Exchange Rate.

Figures 20 (a) and (b) show the real, simulated price using LLGMM, and simulated price using the average parameters $\bar{\beta}$, $\bar{\mu}$, $\bar{\delta}$, $\overline{\sigma}$ and $\bar{\gamma}$ in Table 12 and 13 for U.S. Treasury Bill Yield Interest Rate and U.S. Eurocurrency Exchange Rate respectively.

Chapter 8
Forecasting

8.1 Introduction

In this chapter, we shall sketch an outline about forecasting problem. An $\epsilon-$ sub-optimal simulated value $y^s_{\hat{m}_k,k}$ at time t_k is used to define a forecast $y^f_{\hat{m}_k,k}$ for y_k at the time t_k for each of the Energy commodity model, and the U. S. Interest rate and U.S. Eurocurrency exchange rate.

8.2 Forecasting, Prediction and Confidence Interval for Energy Commodity Model

In the context of Illustration 6.4.1, we begin forecasting from time t_k. Using the data set up to time t_{k-1}, we compute $\hat{m}_i$, $\sigma^2_{\hat{m}_i,i}$, $a_{\hat{m}_i,i}$ and $\mu_{\hat{m}_i,i}$ for $i \in I(0, k-1)$. We assume that we have no information about the real data set $\{y_i\}_{i=k}^N$. Under these considerations, imitating the computational procedure outlined in Section 6.4 and using (6.40), we find the estimate of the forecast $y^f_{\hat{m}_k,k}$ at time t_k as follows;

$$y^f_{\hat{m}_k,k} = y^s_{\hat{m}_{k-1},k-1} + a_{\hat{m}_{k-1},k-1} y^s_{\hat{m}_{k-1},k-1}(\mu_{\hat{m}_{k-1},k-1} - y^s_{\hat{m}_{k-1},k-1})\Delta t + \sigma_{\hat{m}_{k-1},k-1} y^s_{\hat{m}_{k-1},k-1}\Delta W_k, \tag{8.1}$$

where the estimates $\sigma^2_{\hat{m}_{k-1},k-1}$, $a_{\hat{m}_{k-1},k-1}$ and $\mu_{\hat{m}_{k-1},k-1}$ are defined in (6.40) with respect to the known past data set up to the time t_{k-1}. We note that $y^f_{\hat{m}_k,k}$ is the ϵ-sub-optimal estimate for y_k at time t_k .

To determine $y^f_{\hat{m}_{k+1},k+1}$, we need $\sigma^2_{\hat{m}_k,k}$, $a_{\hat{m}_k,k}$ and $\mu_{\hat{m}_k,k}$. Since we only have information of real data up to time t_{k-1}, we use the forecasted estimate $y^f_{\hat{m}_k,k}$ as the estimate of y_k at time t_k, and to estimate $\sigma^2_{\hat{m}_k,k}$, $a_{\hat{m}_k,k}$ and $\mu_{\hat{m}_k,k}$. Hence, we can write $a_{\hat{m}_k,k}$ as $a_{\hat{m}_k,k} \equiv a_{\hat{m}_k, y_{k-\hat{m}_k+1}, y_{k-\hat{m}_k+2},\ldots,y_{k-1},y^f_{m_k,k}}$. We can also re-write $\mu_{\hat{m}_k,k} \equiv \mu_{\hat{m}_k, y_{k-\hat{m}_k+1}, y_{k-\hat{m}_k+2},\ldots,y_{k-1},y^f_{\hat{m}_k,k}}$. To find $y^f_{\hat{m}_{k+2},k+2}$, we use the estimates

$$\begin{aligned}
a_{\hat{m}_{k+1},k+1} &\equiv a_{\hat{m}_{k+1}, y_{k-\hat{m}_k+2}, y_{k-\hat{m}_k+3},\ldots,y_{k-1},y^f_{\hat{m}_k,k},y^f_{\hat{m}_{k+1},k+1}}, \\
\mu_{\hat{m}_{k+1},k+1} &\equiv \mu_{\hat{m}_{k+1}, y_{k-\hat{m}_k+2}, y_{k-\hat{m}_k+3},\ldots,y_{k-1},y^f_{\hat{m}_k,k},y^f_{\hat{m}_{k+1},k+1}}.
\end{aligned}$$

Continuing this process in this manner, we use the estimates

$$\begin{aligned} a_{\hat{m}_{k+i-1},k+i-1} &\equiv a_{\hat{m}_{k+i-1},y_{k-\hat{m}_k+i},y_{k-\hat{m}_k+i+1},\ldots,y_{k-1},y^f_{\hat{m}_k,k},y^f_{\hat{m}_{k+1},k+1},\ldots,y^f_{\hat{m}_{k+1},k+i-1}}, \\ \mu_{\hat{m}_{k+i-1},k+i-1} &\equiv \mu_{\hat{m}_{k+i-1},y_{k-\hat{m}_k+i},y_{k-\hat{m}_k+i+1},\ldots,y_{k-1},y^f_{\hat{m}_k,k},y^f_{\hat{m}_{k+1},k+1},\ldots,y^f_{\hat{m}_{k+1},k+i-1}} \end{aligned}$$

to estimate $y^f_{\hat{m}_{k+i},k+i}$

Prediction/Confidence Interval for Energy Commodities

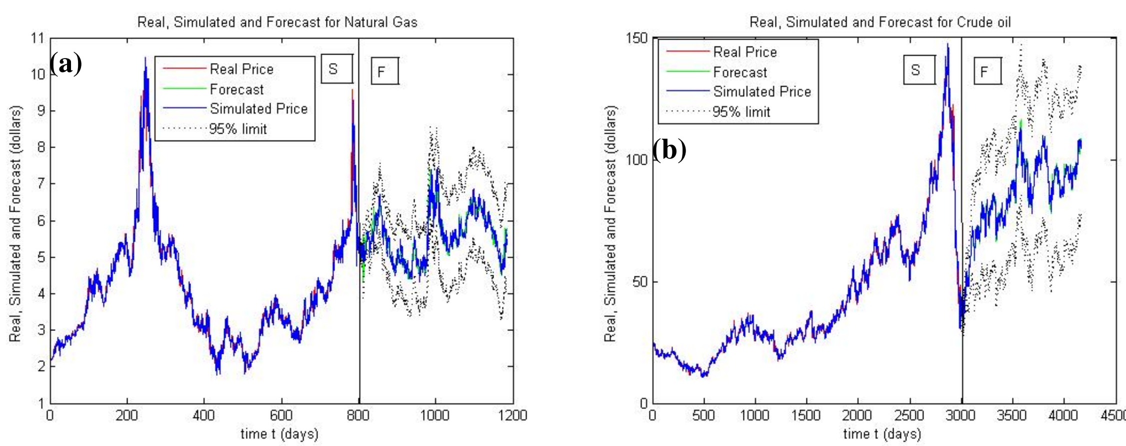
In order to be able to assess the future certainty, we also discuss about the prediction/confidence interval. We define the $100(1-\alpha)\%$ confidence interval for the forecast of the state $y^f_{\hat{m}_i,i}$ at time t_i, $i \geq k$, as $y^f_{\hat{m}_i,i} \pm z_{1-\alpha/2}\left(s^2_{\hat{m}_{i-1},i-1}\right)^{1/2} y^f_{\hat{m}_{i-1},i-1}$, where $\left(s^2_{\hat{m}_{i-1},i-1}\right)^{1/2} y^f_{\hat{m}_{i-1},i-1}$ is the estimate for the sample standard deviation for the forecasted state derived from the following iterative process

$$y^f_{\hat{m}_k,k} = y^f_{\hat{m}_{k-1},k-1} + a_{\hat{m}_{k-1},k-1} y^f_{\hat{m}_{k-1},k-1}(\mu_{\hat{m}_{k-1},k-1} - y^f_{\hat{m}_{k-1},k-1})\Delta t + \sigma_{\hat{m}_{k-1},k-1} y^f_{\hat{m}_{k-1},k-1}\Delta W_k. \tag{8.2}$$

It is clear that the 95 % confidence interval for the forecast at time t_i is

$$\left(y^f_{\hat{m}_i,i} - 1.96\left(s^2_{\hat{m}_{i-1},i-1}\right)^{1/2} y^f_{\hat{m}_{i-1},i-1}, y^f_{\hat{m}_i,i} + 1.96\left(s^2_{\hat{m}_{i-1},i-1}\right)^{1/2} y^f_{\hat{m}_{i-1},i-1}\right)$$

where the lower end denotes the lower bound of the state estimate and the upper end denotes the upper bound of the state estimate.



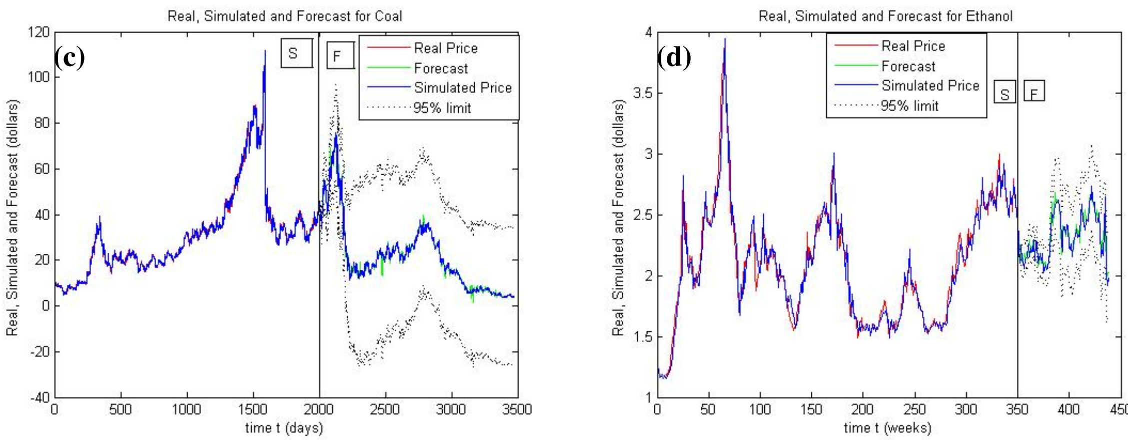
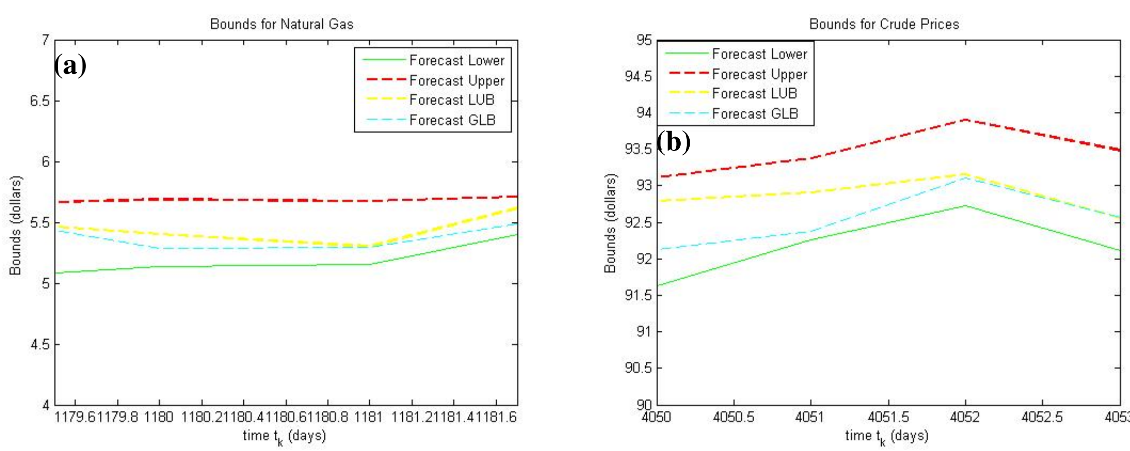


Figure 21.: Real, Simulated and Forecasted Prices for daily Henry Hub natural gas, daily crude oil, daily coal, and weekly ethanol data set.

Figures 21: **(a), (b), (c) and (d)** show the graph of the forecast and 95 percent confidence limit for the daily Henry Hub Natural gas data set [24], daily Crude Oil data set [23], daily Coal data set [22], and weekly Ethanol data set [136], respectively. Figure 21: **(a)**, **(b)**, **(c)** and **(d)** show two region: the simulation region S and the forecast region F. For the simulation region S, we plot the real data set together with the simulated data set as described in Figure 11. For the forecast region F, we plot the estimate of the forecast as explained in Section 8. The upper and the lower simulated sketches in Figure 21 **(a), (b), (c) and (d)** are corresponding to the upper and lower ends of the 95% confidence interval. For details, see Figure 22.

Next, we show a graph of the upper, least upper bound, lower and greatest lower bounds for the estimates of the forecast for the energy commodity processes after running the simulations for 25 times.



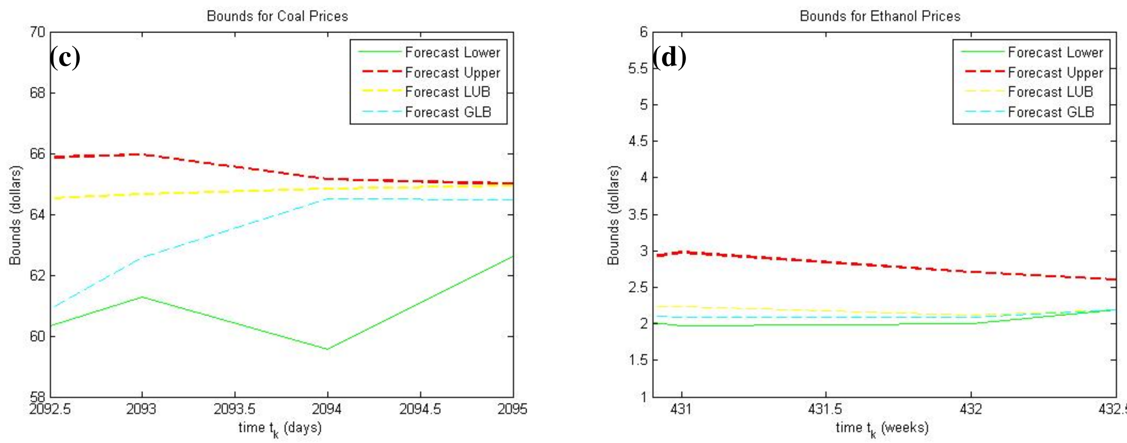


Figure 22.: Bounds for daily Henry Hub natural gas, daily crude oil, daily coal, and weekly ethanol data set.

Figures 22: **(a), (b), (c) and (d)** show the bounds for the daily Henry Hub Natural gas data set [24], daily Crude Oil data set [23], daily Coal data set [22], and weekly Ethanol data set [136], respectively. These bounds are derived after 25 run time (simulations)

8.3 Forecasting and Prediction/Confidence Interval for U. S. Treasury Bill and U. S. Eurocurrency rate

Following the same procedure explained in Section 8.2, we show the graph of the real, simulated, forecast and 95 percent confidence limit for the Interest rate and US dollar Eurocurrency rate.

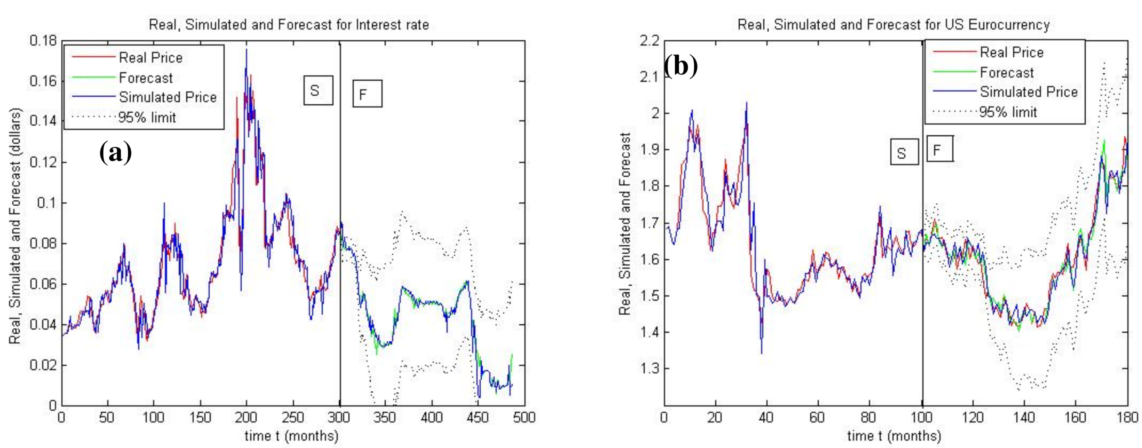


Figure 23.: Real, Simulated, Forecast and 95% Confidence Limit for Interest rate and US Eurocurrency data.

Figure 23(a) shows the real, simulated, forecast and 95 percent confidence limit for the Interest rate data sets and Figure 23(b) shows the real, simulated, forecast and 95 percent confidence limit for the US Eurocurrency data.

Lastly, we show some bounds for the U. S. Treasury Bill Interest Rate and U. S. Eurocurrency rate.

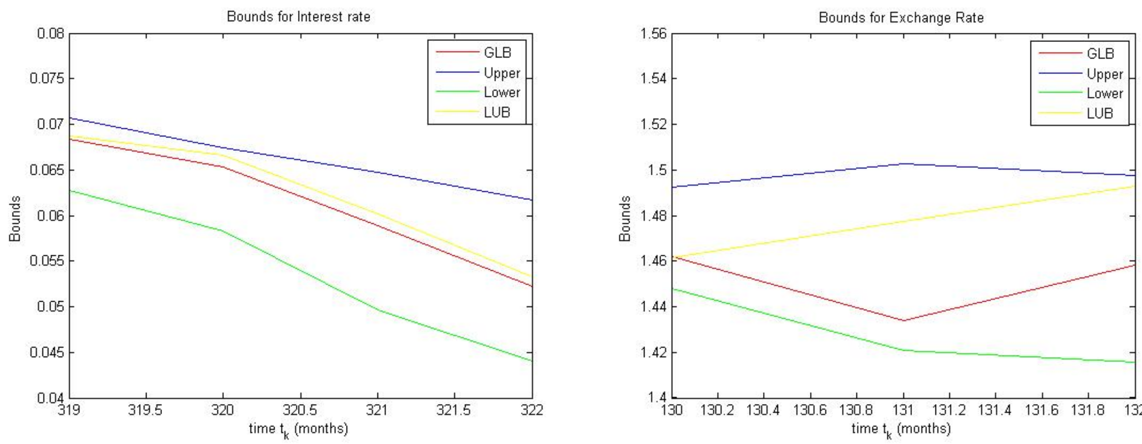


Figure 24.: Bound for U. S. Treasury Bill and U. S. Eurocurrency rate.

Chapter 9

A Two-scale Network Dynamic Model for Energy Commodity Process

9.1 Introduction

Understanding the economy evolution in response to structural changes in energy commodity network system is important to professional economists. The relationship between the different energy sources and their uses provide insights into many important energy issues. The qualitative and quantitative behavior of energy commodities in which the trend in price of one commodity coincides with the trend in prices of other commodities, have always raised the question of whether there is any relationship between prices of energy commodities. If there is any relationship, then what comes to mind is the extent to which one commodity influences the other. Petroleum, natural gas, coal, nuclear fuel, and renewable energy are termed as primary energy components of the energy goods network system because other sources of energy depend on them. Natural gas is usually found near petroleum. This is because of the fact that natural gas and crude oil are rivals in production and substitutes in consumption. As a result of this, energy theory suggests that the two prices should be related. The electric power sector uses primary energy such as coal to generate electricity, which makes electricity a secondary rather than a primary energy source. According to the US Energy Information Administration (EIA), the major energy goods consumed in the United States are petroleum (oil), natural gas, coal, nuclear, and renewable energy. The majority of users are residential and commercial buildings, industry, transportation, and electric power generators. The pattern of fuel usage varies widely by sector [130]. For example, 71% of total petroleum oil provides 93% of the energy used for transportation; 23% of total petroleum oil provides 17% of energy used for residential and commercial use; 5% of total petroleum oil provides 40% of energy used for industrial use; but only 1% of total petroleum oil provides about 1% of the energy used to generate electric power, whereas coal provides 46% of the energy used to generate electric power and natural gas provides 20% of the energy used to generate electric power. This analysis suggests that the strength of interactions between coal and electricity will be stronger than when compared with the strength of interactions between crude oil and electricity, or natural gas and electricity.

Energy price forecasts are highly uncertain. We might expect the price of the natural gas and crude oil to follow the same trend because they are often found mixed with oil in oil wells, and also of the fact that natural gas is often used in petroleum refining and exploration. Recently, Ramberg et al [94] showed that the cointegration relationship between natural gas and crude oil does not appear to be stable through time. They claimed that though there is cointegration between the two energy prices, but there are also recurrent exogeneous factors such as seasonality, episodic heat waves, cold waves and supply interruption from hurricane affecting the trends in the prices. Brown and Y$\ddot{u}$cel [12] also observed that the price of natural gas pulled away from oil prices in 2000, 2002, 2003 and late 2005. Oil prices are influenced by several factors, including some that have mainly short-term impacts and other factors, such as expectations about future world demand for petroleum, other liquids and production decisions of the Organization of the Petroleum Exporting Countries (OPEC) [130]. Supply and demand in the World oil market are balanced through responses to price movement with considerable complexity in the evolution of underlying supply and demand expectation process. For the petroleum and other liquids, the key determinants of long-term supply and prices can be summarized in four broad categories [130]: the economics of non-OPEC supply, OPEC investment and production decisions, the economics of other liquids supply, and World demand for petroleum and other liquids. According to the US Energy Information Administration (EIA) [130] and following the decline of natural gas prices since 2008, real average delivered price for electricity has dropped gradually to 9.8 cents per kilowatthour (kWh) from 2009 to 2012. Retail electricity price is influenced by the fuel price, and particularly by the natural gas price [130]. However, the relationship between retail electricity price and natural gas price is complex. Many factors influence the degree to which and the time frame over which they are linked. A few notable factors are a share of natural gas generation in a region, the level of costs associated with the electricity transmission and distribution systems, the mix of competitive versus cost-of-service pricing, and the number of customers who purchase power directly from wholesale power markets. As a result of this, it can take time for changes in fuel price to affect electricity price. The question that we are now faced is whether the price of electricity depends on the prices of more than one energy commodities, rather than depending on only one commodity (coal or natural gas).

An understanding of how changes in price of one energy commodity are expressed in terms of other energy commodity is needed. This would prove to be useful in predicting price behavior over the long run, and further facilitates profit maximizing process. To check if there is really indeed a relationship between energy commodities; the need to be able to create a model which explains such commodity prices relationship over short and long time interval arises. The relationships between energy commodities have been addressed in [4, 12, 39, 40, 49, 93, 94, 131, 132]. The error correction model [4, 12, 39, 49, 93, 94] is the most commonly used model by authors to describe the relationship between energy commodities. Moreover, Hartley et al [49] have concluded that the U. S. natural gas and crude oil remain linked in their long-term movements. In addition, it is exhibited that there is strong evidence of stable relationship between these two energy commodities which are affected by short run seasonal fluctuations and other factors. The rule of thumb [49] has long been used in the energy industry to relate the natural gas prices to crude oil prices. The rule denoted by the 10-to-1 rule states that the price of natural gas is one tenth of the price of crude oil prices. Similarly, 6-to-1 rule states that the price of natural gas is one sixth of the price of crude oil. It has been examined by Brown et al. [12] that these two rules do not perform well when used to assess the relationship between U.S natural gas price and West Texas Intermediate (WTI) crude oil price for the past 20^{+} years. Moreover, their error correcting model specify the relationship between the two commodities. Using this model, they show that when certain factors are taken into account, movements in crude oil prices can shape the price of natural gas. Vezzoli [131] in his work applies an ordinary least squares (OLS) regression on log of natural gas and crude oil prices. Using this model, he was able to conclude that there is a relationship between natural gas and crude oil prices. Bachmeir et al. [4] showed that the crude oil, coal and natural gas in the United States have weak cross-cointegration using the error correction model. Ramberg et al [94] shows that any simple formula between natural gas and crude oil prices will leave a portion of the natural gas price unexplained. Furthermore, the relationship between natural gas and crude oil using a vector error correction model [12, 94] under the cointegration between the two energy commodities and other factors such as recurrent exogeneous factors are presented. Villar et al. [132] lists some economic factors linking natural gas and crude oil prices, while testing for market integration in the United Kingdom during the time when natural gas was deregulated. Asche et al. [40] have integrated the prices of the energy commodities: natural gas, electricity, and crude oil.

The most of the work is centered around the relationship between prices of energy commodities. In this work, we are interested in an inter-dependence of certain energy commodities. Moreover, we develop a hybrid system of multivariate continuous stochastic network dynamic system.

In this chapter, we further extend the non-linear interconnected stochastic model (4.11) to multivariate interconnected energy commodities and sources with and without external random intervention processes.

9.2 Model Derivation

We denote $\mathbf{p} = [p_1, p_2, ..., p_n]^T$ to be a vector of n energy commodity prices which are considered to have long-run or short-run relationship with each other. Let $p_j(t)$ be the price of the j-th energy commodity at time t. The economic principles of demand and supply processes suggest that the price of a energy commodity will remain within a given finite expected lower and upper bounds. Therefore, $u_j \in \mathbb{R}_+ = (0, \infty)$ and $l_j \geq 0$ stand for the expected upper and lower limits of the j-th energy commodity spot prices, respectively. In the absence of interactions between the energy commodities p_j, $j \in I(1, n)$, where $I(a, b) = \{z \in \mathbb{Z} \mid a \leq z \leq b\}$, the market potential for the jth commodity per unit of time at time t can be characterized by $(u_j - p_j)(l_j + p_j)$. This simple idea leads to the following economic principle regarding the dynamic of the price p_j of the jth energy commodity. The change in spot price of the j-th energy commodity $\Delta p_j(t) = p_j(t + \Delta t) - p_j(t)$ over the interval of length $|\Delta t|$ is directly proportional to the market potential price.

$$\Delta p_j(t) \propto (u_j - p_j)(l_j + p_j)\, \Delta t. \tag{9.1}$$

This implies that

$$dp_j = \alpha_j (u_j - p_j)(l_j + p_j)\, dt, \tag{9.2}$$

for some constant α_j. From this deterministic mathematical model, if $\alpha_j > 0$, we note that as the price falls below the expected price u_j, the price of the jth commodity rises, and as the price lies above u_j, there is a tendency for the price to fall. Similar argument follows if $\alpha_j < 0$. Hence

$$\lim_{t \to \infty} p_j(t) = u_j, \tag{9.3}$$

which implies that u_j is the equilibrium state of (9.2).

In a real World situation, the expected upper price limit u_j of the j-th commodity is not a constant parameter. It varies with time, and moreover it is subject to random environmental perturbations. Therefore, we consider

$$u_j = y_j + \xi_j, \tag{9.4}$$

where ξ_j is a white noise process that characterizes the measure of random fluctuation of the upper price limit of the j-th commodity; here y_j stands for the mean of the energy spot price process of the j-th commodity at time t. It is further assumed that y_j is governed by a similar dynamic forces described in (9.2), that is,

$$dy_j = \mu_j (u_j - y_j)(v_j + y_j)\, dt, \tag{9.5}$$

where $\mu_j > 0$ is defined as the mean reversion rate of the mean of the j-th commodity, $\upsilon_j \geq 0$ is defined as the lower limit of the mean of the j-th commodity. By following the argument used in (9.4), we incorporate the effects of random environmental perturbations into the lower limit υ_j of the mean of the j-th commodity:

$$\upsilon_j = v_j + e_j, \tag{9.6}$$

where $v_j \geq 0$, and e_j is a white noise process describing the measure of random influence on the mean price of the j-th commodity.

Substituting expressions in (9.4) and (9.6) into (9.2) and (9.5), respectively, we obtain

$$\begin{cases} dy_j &= \mu_j \left(u_j - y_j\right)\left(v_j + y_j\right)dt + \mu_j \left(u_j - y_j\right) e_j(t)dt \\ dp_j &= \alpha_j \left(y_j - p_j\right)\left(l_j + p_j\right)dt + \alpha_j \left(l_j + p_j\right)\xi_j(t)dt. \end{cases} \tag{9.7}$$

In the absence of interactions and using (9.7), the system of stochastic model for isolated expected spot and spot prices processes are described by the following non-linear system of stochastic differential equations:

$$\begin{cases} dy_j &= \mu_j \left(u_j - y_j\right)\left(v_j + y_j\right)dt + \delta_{j,j}\left(u_j - y_j\right)dW_{j,j}(t), \;\; y_j(t_0) = y_{j0}, \\ dp_j &= \alpha_j \left(y_j - p_j\right)\left(l_j + p_j\right)dt + \sigma_{j,j}\left(l_j + p_j\right)dZ_{j,j}(t), \; p_j(t_0) = p_{j0}, \; j \in I(1,n), \end{cases} \tag{9.8}$$

where

$$\begin{cases} \mu_j e_j(t)dt = \delta_{j,j} dW_{j,j}(t), \; \text{j=1,2,...,n}, \\ \alpha_j \xi_j(t)dt = \sigma_{j,j} dZ_{j,j}(t), \; \text{j=1,2,...,n}, \end{cases}$$

and $\delta_{j,j}, \sigma_{i,j}$ are non-negative for $j = 1, 2, ..., n$.

In the presence of interactions, for each $j \in I(1,n)$, we consider both deterministic and stochastic interaction functions. For each $j \in I(1,n)$, we define the j-th aggregate interaction functions $\mathbf{g}_j : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ and $\boldsymbol{h}_j : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ for the jth commodity of the mean energy spot price process $\mathbf{y}_j(t)$ and the energy spot price process $\mathbf{p}_j(t)$ in a energy commodity market network system, respectively. Moreover, we assume that these functions have the following structural forms;

$$\begin{cases} \mathbf{g}_j(t, \mathbf{y}) = \mathbf{g}_j(t, k_{j,1}y_1, k_{j,2}y_2, ..., k_{j,n}y_n) \\ \boldsymbol{h}_j(t, \mathbf{p}) = \boldsymbol{h}_j(t, \gamma_{j,1}p_1, \gamma_{j,2}p_2, ..., \gamma_{j,n}p_n), \end{cases} \tag{9.9}$$

where $k_{j,i}$ and $\gamma_{j,i}$ are elements of the $n \times n$ interconnection matrices $\mathbf{E}_g$ and $\mathbf{E}_h$, respectively. In (9.9), $k_{j,i}$ and $\gamma_{j,i} : \mathbb{R}_+ \to [0,1]$ represent a degree of interaction of the j-th commodity with i-th commodity in the energy commodity market network system.

For the matrix $\mathbf{E}_g$, $k_{j,i} = 0$ with fixed $i \in I(1,n)$ if the i-th commodity in the energy market network system does not influence the j-th commodity. Likewise, for the matrix $\mathbf{E}_h$, $\gamma_{j,i} = 0$ with fixed $i \in I(1,n)$, the j-th commodity in the energy market network system sub-component of $\mathbf{p}$ is totally unaffected by the influence of the i-th commodity.

Finally, we introduce interactions in the diffusion coefficients with respect to the j-th commodity of the energy market network system under random environmental perturbations as: $\boldsymbol{\psi}_j : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ and $\boldsymbol{\Lambda}_j : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ for each $j \in I(1,n)$. The diffusion part is of the form

$$\begin{cases} \boldsymbol{\psi}_j(t, \mathbf{y}) \cdot \mathbf{e}_j(t)dt &= \sum\limits_{l=1}^{n} \boldsymbol{\psi}_{j,l}(t, y_l)dW_{j,l}(t) \\ \boldsymbol{\Lambda}_j(t, \mathbf{p}) \cdot \boldsymbol{\xi}_j(t)dt &= \sum\limits_{l=1}^{n} \boldsymbol{\Lambda}_{j,l}(t, p_l)dZ_{j,l}(t), \end{cases} \tag{9.10}$$

where $\mathbf{e}_j$ and $\boldsymbol{\xi}_j$ are n-dimensional white noise processes; $\cdot$ stands for dot product.

We assume that the interaction functions (9.9) and (9.10) have the following forms;

$$\begin{cases} \mathbf{g}(t, \mathbf{y}) &= \boldsymbol{\gamma}(t, \mathbf{y})\mathbf{G}(t, \mathbf{y}) \\ \mathbf{h}(t, \mathbf{p}) &= \boldsymbol{\lambda}(t, \mathbf{p})\mathbf{H}(t, \mathbf{p}), \\ \boldsymbol{\psi}(t, \mathbf{y}) &= \boldsymbol{\gamma}(t, \mathbf{y})\boldsymbol{\Psi}(t, \mathbf{y}), \\ \boldsymbol{\Lambda}(t, \mathbf{p}) &= \boldsymbol{\lambda}(t, \mathbf{p})\boldsymbol{\Phi}(t, \mathbf{p}), \end{cases}$$

where $\mathbf{g}(t, \mathbf{y}) = [\mathbf{g}_1(t, \mathbf{y}), ..., \mathbf{g}_j(t, \mathbf{y}), ..., \mathbf{g}_n(t, \mathbf{y})]^T$, $\mathbf{h}(t, \mathbf{p}) = [\mathbf{h}_1(t, \mathbf{p}), ..., \mathbf{h}_j(t, \mathbf{p}), ..., \mathbf{h}_n(t, \mathbf{p})]^T$ are defined in (9.9), $\boldsymbol{\psi}(t, \mathbf{y}) = \left(\boldsymbol{\psi}_{j,l}(t, \mathbf{y})\right)_{n \times n}$, and $\boldsymbol{\Lambda}(t, \mathbf{p}) = (\boldsymbol{\Lambda}_{j,l}(t, \mathbf{p}))_{n \times n}$, $\boldsymbol{\gamma}(t, \mathbf{y}) = diag(u_1 - y_1, ..., u_j - y_j, ..., u_n - y_n)$ and $\boldsymbol{\lambda}(t, \mathbf{p}) = diag(l_1 + p_1, ..., l_j + p_j, ..., l_n + p_n)$; $\boldsymbol{G}$, and $\boldsymbol{H}$ are $n \times 1$ column vectors; $\boldsymbol{\Psi} = diag(\boldsymbol{\psi}_1, ..., \boldsymbol{\psi}_j, ..., \boldsymbol{\psi}_n)$ and $\boldsymbol{\Phi} = diag(\boldsymbol{\Lambda}_1, ..., \boldsymbol{\Lambda}_j, ..., \boldsymbol{\Lambda}_n)$ are block diagonal matrices; $\boldsymbol{\psi}_j = [\boldsymbol{\psi}_{j,1}, ..., \boldsymbol{\psi}_{j,l}, ..., \boldsymbol{\psi}_{j,n}]$, $\boldsymbol{\Lambda}_j = [\boldsymbol{\Lambda}_{j,1}, ..., \boldsymbol{\Lambda}_{j,l}, ..., \boldsymbol{\Lambda}_{j,n}]$. We also assume that $\mathbf{G}$, $\mathbf{H}$, $\boldsymbol{\Psi}$ and $\boldsymbol{\Phi}$ satisfy the local Lipschitz condition. This assumption implies that $\mathbf{g}$, $\mathbf{h}$, ψ and $\boldsymbol{\Lambda}$ also satisfy local Lipschitz condition.

Thus, the interconnected energy commodity network system is described by

$$\begin{cases} dy_j &= (u_j - y_j)\left[\left(\mu_j\left(v_j + y_j\right) + \mathbf{G}_j(t, \mathbf{y})\right)dt + \delta_{j,j}dW_{j,j}(t) + \sum\limits_{l=1}^{n} \boldsymbol{\Psi}_{j,l}(t, \mathbf{y})dW_{j,l}(t)\right], \\ & y_j(t_0) = y_{j0}, \\ dp_j &= (l_j + p_j)\left[\left(\alpha_j\left(y_j - p_j\right) + \boldsymbol{H}_j(t, \mathbf{p})\right)dt + \sigma_{j,j}dZ_{j,j}(t) + \sum\limits_{l=1}^{n} \boldsymbol{\Phi}_{j,l}(t, \mathbf{p})dZ_{j,l}(t)\right], \\ & p_j(t_0) = p_{j0},\ j \in I(1,n), \end{cases} \tag{9.11}$$

where the parameters $\mu_j > 0$; $\alpha_j > 0$; $u_j > 0$; $v_j \geq 0$; $l_j \geq 0$; $\delta_{j,j} > 0$; $\sigma_{j,j} > 0$; and for $j \neq l$, $\delta_{j,l} \geq 0$; $\sigma_{j,l} \geq 0$; $j, l \in I(1,n)$; for $j \in I(1,n)$, $\mathbf{W}_j$ and $\mathbf{Z}_j$ are n-dimensional independent Wiener processes defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathcal{P})$; for $l \neq i$, $\mathbb{E}[dW_{j,l}dW_{k,i}] = 0$, and for $l = i$, $\mathbb{E}[dW_{j,l}dW_{k,i}] = dt$; the filtration function $(\mathcal{F}_t)_{t\geq 0}$ is right-continuous; for each $t \geq 0$, each $\mathcal{F}_t$ contains all $\mathcal{P}$-null sets in $\mathcal{F}$; the n-dimensional random vectors $\mathbf{y}(t_0)$ and $\mathbf{p}(t_0)$ are $\mathcal{F}_{t_0}$ measurable.

The network system of stochastic differential equations in (9.11) can be written as follows;

$$\begin{cases} d\mathbf{y} = \mathbf{a}(t,\mathbf{y})dt + \boldsymbol{\Upsilon}(t,\mathbf{y})d\mathbf{W}(t), \ \mathbf{y}(t_0) = \mathbf{y}_0 \\ d\mathbf{p} = \mathbf{b}(t,\mathbf{y},\mathbf{p})dt + \boldsymbol{\sigma}(t,\mathbf{p})d\mathbf{Z}(t), \ \mathbf{p}(t_0) = \mathbf{p}_0, \end{cases} \tag{9.12}$$

where

$$\begin{cases} \mathbf{a}(t,\mathbf{y}) & = & \begin{pmatrix} (u_1 - y_1)\left[\mu_1\left(v_1 + y_1\right) + \mathbf{G}_1(t,\mathbf{y})\right] \\ (u_2 - y_2)\left[\mu_2\left(v_2 + y_2\right) + \mathbf{G}_2(t,\mathbf{y})\right] \\ \vdots \\ (u_n - y_n)\left[\mu_n\left(v_n + y_n\right) + \mathbf{G}_n(t,\mathbf{y})\right] \end{pmatrix}, \\ \mathbf{b}(t,\mathbf{y},\mathbf{p}) & = & \begin{pmatrix} (l_1 + p_1)\left[\alpha_1\left(y_1 - p_1\right) + \boldsymbol{H}_1(t,\mathbf{p})\right] \\ (l_2 + p_2)\left[\alpha_2\left(y_2 - p_2\right) + \boldsymbol{H}_2(t,\mathbf{p})\right] \\ \vdots \\ (l_n + p_n)\left[\alpha_n\left(y_n - p_n\right) + \boldsymbol{H}_n(t,\mathbf{p})\right] \end{pmatrix}, \\ \boldsymbol{\Upsilon}(t,\mathbf{y}) & = & diag(A_1(\mathbf{y}),...,A_j(\mathbf{y}),..,A_n(\mathbf{y})), \qquad \boldsymbol{\sigma}(t,\mathbf{p}) = diag(B_1(\mathbf{p}),...,B_j(\mathbf{p}),...,B_n(\mathbf{p})), \end{cases}$$

and

$$\begin{cases} \mathbf{A}_j(\mathbf{y}) & = & (u_j - y_j)\begin{pmatrix} \boldsymbol{\Psi}_{j,1} & \boldsymbol{\Psi}_{j,2} & \ldots & \boldsymbol{\Psi}_{j,j-1} & \delta_{j,j} + \boldsymbol{\Psi}_{j,j} & \boldsymbol{\Psi}_{j,j+1} & \ldots & \boldsymbol{\Psi}_{j,n} \end{pmatrix}, \\ \mathbf{B}_j(\mathbf{p}) & = & (l_j + p_j)\begin{pmatrix} \boldsymbol{\Phi}_{j,1} & \boldsymbol{\Phi}_{j,2} & \ldots & \boldsymbol{\Phi}_{j,j-1} & \sigma_{j,j} + \boldsymbol{\Phi}_{j,j} & \boldsymbol{\Phi}_{j,j+1} & \ldots & \boldsymbol{\Phi}_{j,n} \end{pmatrix}; \end{cases}$$

$\mathbf{W} = [W_1, ...W_j, ..., W_n]^T$, and $\mathbf{Z} = [Z_1, ...Z_j, ..., Z_n]^T$ are block matrices; $W_j = [W_{j,1}, ...W_{j,2}, ..., W_{j,n}]^T$, $Z_j = [Z_{j,1}, ...Z_{j,2}, ..., Z_{j,n}]^T$; and $\boldsymbol{\Upsilon}(t,\mathbf{y})$, $\boldsymbol{\sigma}(t,\mathbf{p})$ are a $n \times n$ block matrix with each entries having order $1 \times n$.

In the next section, we outline the model validation problems of (9.12), namely, the existence and uniqueness of solution process.

9.3 Mathematical Model Validation

In this section, we validate the mathematical model derived in Section 2. We note that the classical existence and uniqueness theorem [57, 66] is not directly applicable to (9.12). We need to modify the existence and uniqueness results. The modification is based on Theorem 3.4 [57]. We show the global existence of solution process of system of differential equations (9.12).

We note that the rate functions $\mathbf{a}$, $\mathbf{b}$, $\boldsymbol{\Upsilon}$, and $\boldsymbol{\sigma}$ in stochastic system of differential equations (9.12) do not satisfy the classical existence and uniqueness conditions [57]. However these rate functions do satisfy the local Lipschitz condition. Therefore, we construct sequences of functions for the drift and diffusion coefficients of interconnected dynamic system (9.12) so that the classical conditions for existence and uniqueness theorem are applicable. The construction of modification scheme is as follows: First, we define a cylindrical subset $[t_0,\infty) \times U_m$ of $[0,\infty)\times\mathbb{R}^n$ for $t_0 \in [0,\infty)$, $m \in I(1,\infty)$, where U_m is an n- dimensional sphere with radius m defined by

$$U_m = \mathbb{B}(\mathbf{x}_0, m) = \{\mathbf{x} \in \mathbb{R}^n \ : \ ||\mathbf{x} - \mathbf{x}_0|| < m\},$$

for any $m \in I(1,\infty)$. We note that U_m is inscribed in n-dimensional parallelepiped $\mathbb{R}(\mathbf{x}-\mathbf{x}_0, m) = [-m,m] \times ... \times [-m,m]$ in $\mathbb{R}^n$.

The developed stochastic network model (9.12) can be written as:

$$\begin{cases} d\mathbf{y} = \mathbf{a}^m(t,\mathbf{y})dt + \boldsymbol{\Upsilon}^m(t,\mathbf{y})d\mathbf{W}(t), \ \mathbf{y}(t_0) = \mathbf{y}_0 \\ d\mathbf{p} = \mathbf{b}^m(t,\mathbf{y},\mathbf{p})dt + \boldsymbol{\sigma}^m(t,\mathbf{p})d\mathbf{Z}(t), \ \mathbf{p}(t_0) = \mathbf{p}_0, \end{cases} \tag{9.13}$$

where

$$\begin{cases} \mathbf{a}^m(t,\mathbf{y}) &= \mathbf{a}(t,\mathbf{q}(\mathbf{y},m)) \\ \boldsymbol{\Upsilon}^m(t,\mathbf{y}) &= \boldsymbol{\Upsilon}(t,\mathbf{q}(\mathbf{y},m)), \\ \mathbf{b}^m(t,\mathbf{y},\mathbf{p}) &= \mathbf{b}(t,\mathbf{q}(\mathbf{y},m),\mathbf{q}(\mathbf{p},m)), \\ \boldsymbol{\sigma}^m(t,\mathbf{p}) &= \boldsymbol{\sigma}(t,\mathbf{q}(\mathbf{p},m)). \end{cases} \tag{9.14}$$

Here, for each $j \in I(1,n)$ and $\mathbf{x} \in \mathbb{R}^n$, we define $\mathbf{q}_j(\mathbf{x},m) = \max\{-m, \min\{x_j - x_{0j}, m\}\}$. Hence, $\mathbf{q}(\mathbf{x},m) = [\mathbf{q}_1(\mathbf{x},m), ..., \mathbf{q}_j(\mathbf{x},m), ..., \mathbf{q}_n(\mathbf{x},m)]^T$, and it is denoted by $\mathbf{x}^{(m)}$.

REMARK 23 We observe that $\mathbf{q}(\mathbf{x},m)$ satisfies global Lipschitz condition on $\mathbb{R}^n$ with a Lipschitz constant 1. This together with the local Lipschitz condition assumption on the drift and diffusion coefficients of network system of stochastic differential equations (9.12), the modified rate coefficient functions in (9.13) satisfy the classical existence and uniqueness conditions [57, 66]. Thus, its solution is denoted by $(\mathbf{y}_m, \mathbf{p}_m)$, for $m \in I(1, \infty)$. Moreover, it is assumed that $(\mathbf{y}, \mathbf{p})$ is non-negative whenever $\mathbf{y}_0$, $\mathbf{p}_0 \in \mathbb{R}^n_+$.

Now we apply Theorems 3.4 and 3.5 of [57] in the context of modified system of stochastic differential equations (9.13) and Remark 23 to establish the global existence of solution of stochastic differential equations in (9.13). For this purpose, we outline the argument used in the proof of these theorems.

In addition to the local Lipschitz conditions on the drift and diffusion coefficients, we further impose the following hypothesis on the coefficients:

$(\mathbf{H}_1)$

$$\begin{cases} |\mathbf{g}_j(t,\mathbf{y})| & \leq \quad a_{1,j} + \kappa_j \|\mathbf{y}\|, \\ |\boldsymbol{h}_j(t,\mathbf{p})| & \leq \quad a'_{1,j} + \gamma_j \|\mathbf{p}\|, \\ |\boldsymbol{\psi}_{j,l}(t,\mathbf{y})| & \leq \quad a_{2,j} + \tilde{\delta}_{j,l} \|\mathbf{y}\|, \\ |\boldsymbol{\Lambda}_{j,l}(t,\mathbf{p})| & \leq \quad a'_{2,j} + \tilde{\sigma}_{j,l} \|\mathbf{p}\|. \end{cases} \tag{9.15}$$

where for $i \in I(1,2)$, $a_{i,j}$, $a'_{i,j}$ are non-negative; κ_j, γ_j, $\tilde{\delta}_{j,l}, \tilde{\sigma}_{j,l} \in \mathbb{R}_+$. From (9.13), we further remark that dynamic of mean of spot price vector **y** is decoupled with the dynamic of spot price **p**. Now we first apply Theorems 3.4 and 3.5 of [57] in the context of modified system of stochastic differential equations (9.13) and hypothesis $(\mathbf{H}_1)$ to establish the global existence of solution of the completely decoupled sub-system of stochastic differential equations in (9.13). For this purpose, we outline the argument used in the proof of these theorems.

DEFINITION 9.3.1 *Let $\boldsymbol{\tau}_m$ be the first exit time of the solution process $\boldsymbol{y}_m$ from the set $\mathbb{B}(\boldsymbol{y}_0, m)$. Define $\boldsymbol{\tau}$ to be the (finite or infinite) limit of the monotone increasing sequence $\boldsymbol{\tau}_m$ as $m \to \infty$.*

$$\boldsymbol{\tau} = \lim_{m \to \infty} \boldsymbol{\tau}_m.$$

We wish to show that

$$\mathcal{P}(\boldsymbol{\tau} = \infty) = 1. \tag{9.16}$$

In the following, we present a result that is parallel to Theorem 3.5 [57] in the context of the completely decoupled sub-system of stochastic differential equation (9.12). For this purpose, it is enough to exhibit the global existence result for the transformed system (9.13).

LEMMA 9.1 *For $m \in I(1,\infty)$, and $\mathbf{y}_0 \in \mathbb{R}^n_+$, let $\mathbf{y}_m(t) = \mathbf{y}_m(t,t_0,\mathbf{y}_0)$ be the solution of the completely decoupled sub-system of (9.13), and let the hypothesis ($\boldsymbol{H}_1$) be satisfied. Let V_1 be a function defined on $[t_0,\infty)\times\mathbb{R}^n_+$ into $\mathbb{R}_+$, it is defined by;*

$$V_1(t,\mathbf{y}) = \ln(||\mathbf{y}||^2 + e), \tag{9.17}$$

Then there exists some constant $\mathbf{c}_1 > 0$ such that

$$\begin{cases} \boldsymbol{L}V_1 \le \mathbf{c}_1 V_1 \\ V_{1,m} = \inf_{||\mathbf{y}||>m} V_1(t,\mathbf{y}) \to \infty \ \ as \ \ m \to \infty, \end{cases} \tag{9.18}$$

where $\boldsymbol{L}$ is the differential operator with respect to (9.12); $e = \exp(1)$.

Moreover, the global existence of solution of the completely decoupled sub-system of (9.12) follows.

Proof.

It is obvious that $V_1 \in \mathcal{C}_{1,2}$ on $[t_0,\infty)\times\mathbb{R}^n_+ \to \mathbb{R}_+$. In fact, $\frac{\partial V_1(t,\mathbf{y})}{\partial y_j} = \frac{2y_j}{(||\mathbf{y}||^2+e)}$, $\frac{\partial^2 V_1(t,\mathbf{y})}{\partial y_j^2} = \frac{2}{(||\mathbf{y}||^2+e)} - \frac{4y_j^2}{(||\mathbf{y}||^2+e)^2}$, $\frac{\partial^2 V_1(t,\mathbf{y})}{\partial y_i y_j} = -\frac{4y_iy_j}{(||\mathbf{y}||^2+e)^2}$ exist and are continuous functions defined on $[t_0,\infty)\times\mathbb{R}^n_+ \to \mathbb{R}$. Moreover, the $\mathbf{L}$-operator with respect to the completely decoupled component is as follows:

$$\begin{aligned}
\mathbf{L}V_1(t,\mathbf{y}) \;=\; & \sum_{j=1}^n \left[\mu_j(u_j-y_j)(v_j+y_j) + \mathbf{g}_j(t,\mathbf{y})\right]\frac{\partial V_1(t,y)}{\partial y_j} \\
& + \frac{1}{2}\sum_{j=1}^n \left[[\delta_{j,j}(u_j-y_j) + \boldsymbol{\psi}_{jj}(t,\mathbf{y})]^2 + \sum_{l\neq j}^n \boldsymbol{\psi}^2_{j,l}(t,\mathbf{y})\right]\frac{\partial^2 V_1(t,y)}{\partial y_j^2} \\
& + \frac{1}{2}\sum_{i=1}^n\sum_{\substack{j=1\\ j\neq i}}^n \left[\sum_{l\neq i,j}^n \boldsymbol{\psi}_{i,l}(t,\mathbf{y})\boldsymbol{\psi}_{j,l}(t,\mathbf{y}) + 2[\delta_{i,i}(u_i-y_i)+\boldsymbol{\psi}_{i,i}]\boldsymbol{\psi}_{j,i}\right]\frac{\partial^2 V_1(t,y)}{\partial y_i y_j} \\
=\; & \sum_{j=1}^n \mu_j\left(-\left[y_j - \left(\frac{u_j-v_j}{2}\right)\right]^2 + \left(\frac{u_j+v_j}{2}\right)^2\right)\frac{2y_j}{(||\mathbf{y}||^2+e)} + \sum_{j=1}^n \frac{2\mathbf{g}_j(t,\mathbf{y})y_j}{(||\mathbf{y}||^2+e)} \\
& + \frac{1}{2}\sum_{j=1}^n [\delta_{j,j}(u_j-y_j)+\boldsymbol{\psi}_{j,j}]^2\left(\frac{2}{(||\mathbf{y}||^2+e)} - \frac{4y_j^2}{(||\mathbf{y}||^2+e)^2}\right) \\
& + \frac{1}{2}\sum_{j=1}^n\sum_{l\neq j}^n \boldsymbol{\psi}^2_{j,l}(t,\mathbf{y})\left(\frac{2}{(||\mathbf{y}||^2+e)} - \frac{4y_j^2}{(||\mathbf{y}||^2+e)^2}\right)
\end{aligned}$$

$$
\begin{aligned}
& -\frac{1}{2}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\left[\sum_{l\neq i,j}^{n}\boldsymbol{\psi}_{i,l}(t,y_l)\boldsymbol{\psi}_{j,l}(t,\mathbf{y})+2[\delta_{i,i}(u_i-y_i)+\boldsymbol{\psi}_{i,i}]\boldsymbol{\psi}_{j,i}\right]\frac{4y_iy_j}{(||\mathbf{y}||^2+e)^2}\\
\leq\ & 2\sum_{j=1}^{n}\mu_j\left(\frac{u_j+v_j}{2}\right)^2\frac{y_j}{(||\mathbf{y}||^2+e)}+\sum_{j=1}^{n}\frac{2\mathbf{g}_j(t,\mathbf{y})y_j}{(||\mathbf{y}||^2+e)}+\sum_{j=1}^{n}\frac{[\delta_{j,j}(u_j-y_j)+\boldsymbol{\psi}_{jj}(t,\mathbf{y})]^2}{(||\mathbf{y}||^2+e)}\\
& +\sum_{j=1}^{n}\sum_{l\neq j}^{n}\frac{\boldsymbol{\psi}_{j,l}^2(t,\mathbf{y})}{(||\mathbf{y}||^2+e)}\\
& -\frac{1}{2}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\left[\sum_{l\neq i,j}^{n}\boldsymbol{\psi}_{i,l}(t,y_l)\boldsymbol{\psi}_{j,l}(t,\mathbf{y})+2[\delta_{i,i}(u_i-y_i)+\boldsymbol{\psi}_{i,i}]\boldsymbol{\psi}_{j,i}\right]\frac{4y_iy_j}{(||\mathbf{y}||^2+e)^2}\\
\leq\ & 2\sum_{j=1}^{n}\mu_j\left(\frac{u_j+v_j}{2}\right)^2+\sum_{j=1}^{n}\frac{(a_{1,j}+\kappa_j||\mathbf{y}||)^2+y_j^2}{(||\mathbf{y}||^2+e)}\\
& +2\sum_{j=1}^{n}\left[\delta_{j,j}^2(u_j+1)^2+(a_{2,j}+\tilde{\delta}_{j,j})^2\right]+\sum_{j=1}^{n}\sum_{l\neq j}^{n}(a_{2,j}+\tilde{\delta}_{j,l})^2\\
& +2\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{l\neq i,j}^{n}\left(a_{2,i}+\tilde{\delta}_{i,l}\right)\left(a_{2,j}+\tilde{\delta}_{j,l}\right)\\
& +4\sum_{i=1}^{n}\sum_{j\neq i}^{n}(a_{2,j}+\tilde{\delta}_{j,i})\left[\delta_{i,i}(u_i+1)+(a_{2,i}+\tilde{\delta}_{i,i})\right]\\
\leq\ & c_1V_1(t,\mathbf{y}),
\end{aligned}
$$

where $c_1 = 1+2\sum_{j=1}^{n}\mu_j\left(\frac{u_j+v_j}{2}\right)^2+\sum_{j=1}^{n}(a_{1,j}+\kappa_j)^2+2\sum_{j=1}^{n}\left[\delta_{j,j}^2(u_j+1)^2+(a_{2,j}+\tilde{\delta}_{j,j})^2\right]+\sum_{j=1}^{n}\sum_{l\neq j}^{n}(a_{2,j}+\tilde{\delta}_{j,l})^2+2\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{l\neq i,j}^{n}(a_{2,i}+\tilde{\delta}_{i,l})\left(a_{2,j}+\tilde{\delta}_{j,l}\right)$
$+\,4\sum_{i=1}^{n}\sum_{j\neq i}^{n}(a_{2,j}+\tilde{\delta}_{j,i})\left[\delta_{i,i}(u_i+1)+(a_{2,i}+\tilde{\delta}_{i,i})\right]$

Furthermore, $\inf\limits_{||\mathbf{y}||>m} V_1(t,\mathbf{y})\to\infty$ as $m\to\infty$.

To show that $\mathcal{P}(\boldsymbol{\tau}=\infty)=1$, we define a function

$$
V(t,\mathbf{y})=V_1(t,\mathbf{y})\exp\{-\mathbf{c}_1(t-t_0)\}. \tag{9.19}
$$

It is obvious that $\mathbf{L}V\leq 0$. By defining $\tau_m(t)=\min(\tau_m,t)$; $\mathcal{Y}(t)=\mathbf{y}_m(t)$ for $t<\tau_m$; and imitating the argument of Lemma 3.2 [57], we have

$$
\mathbf{E}\{V_1(\tau_m(t),\mathcal{Y}(\tau_m(t)))\leq e^{\mathbf{c}_1(t-t_0)}\mathbf{E}V_1(t_0,\mathbf{y}(t_0)).
$$

Hence

$$
\mathbf{P}\{\tau_m\leq t\}\leq\frac{e^{\mathbf{c}_1(t-t_0)}\mathbf{E}V_1(t_0,\mathcal{Y}(t_0))}{\inf\limits_{||\mathbf{y}||>m,u>t_0}V_1(u,\mathbf{y})}\to 0\ \text{ as }\ m\to\infty\ \text{ by }\ (9.18). \tag{9.20}
$$

The global existence and uniqueness of solution of the first component of (9.13) follows by letting $m \to \infty$. Hence, from this and the fact that the solution process of transformed system (9.13) is almost surely identical with the solution process of the original system (9.12), we conclude the global existence and uniqueness of (9.12).

□

Now by following the idea of Lemma 9.1, we present a global existence and uniqueness of solution of the system of stochastic differential equations governing the sub-system **p** in (9.12). We simply state a Lemma without the full proof.

LEMMA 9.2 *For $m \in I(1,\infty)$, and $\boldsymbol{y}_0$, $\boldsymbol{p}_0 \in \mathbb{R}^n_+$, let $\boldsymbol{p}_m(t) = \boldsymbol{p}_m(t, t_0, \boldsymbol{p}_0)$ be the solution of the system of stochastic differential equations governing the sub-system **p** described in (9.13), and let the hypothesis ($\boldsymbol{H}_1$) be satisfied. Let V_2 be a nonnegative function on $[t_0,\infty) \times \mathbb{R}^n_+$ into $\mathbb{R}_+$ defined by;*

$$V_2(t,\boldsymbol{p}) = \ln(||\boldsymbol{p}||^2 + e) + \sum_{j=1}^{n} \frac{\alpha_j}{2} \int_t^T (y_j(s) + l_j)^2 \, ds, \tag{9.21}$$

Then there exist a constant $c > 0$ such that

$$\begin{cases} \boldsymbol{L}V_2 \leq cV_2 \\ V_{2,m} = \inf\limits_{||\boldsymbol{p}||>m} V_2(t,\boldsymbol{p}) \to \infty \ \ as \ \ m \to \infty. \end{cases} \tag{9.22}$$

*where **L** is the differential operator with respect to (9.12); $e = \exp(1)$.*

*Moreover, the global existence of solution of the system of stochastic differential equations governing the sub-system **p** described in (9.12) follows.*

Proof. It is obvious that $V_2 \in \mathcal{C}_{1,2}$ on $[t_0,\infty) \times \mathbb{R}^n_+ \to \mathbb{R}_+$. In fact, $\frac{\partial V_2(t,\mathbf{p})}{\partial t} = -\sum\limits_{j=1}^{n} \frac{\alpha_j}{2} (y_j(t) + l_j)^2$, $\frac{\partial V_2(t,\mathbf{p})}{\partial p_j} = \frac{2p_j}{(||\mathbf{p}||^2+e)}$, $\frac{\partial^2 V_2(t,\mathbf{p})}{\partial p_j^2} = \frac{2}{(||\mathbf{p}||^2+e)} - \frac{4p_j^2}{(||\mathbf{p}||^2+e)^2}$, $\frac{\partial^2 V_2(t,\mathbf{p})}{\partial p_i p_j} = -\frac{4p_ip_j}{(||\mathbf{p}||^2+e)^2}$ exist and are continuous functions defined on $[t_0,\infty) \times \mathbb{R}^n_+ \to \mathbb{R}$. Moreover, the **L**-operator with respect to the system of stochastic differential equations governing the sub-system **p** described in (9.12) is as follows:

$$
\begin{aligned}
\mathbf{L}V_2(t,\mathbf{p}) &= -\sum_{j=1}^{n}\frac{\alpha_j}{2}\left(y_j(t)+l_j\right)^2+\sum_{j=1}^{n}\left[\alpha_j(y_j-p_j)(l_j+p_j)+\boldsymbol{h}_j(t,\mathbf{p})\right]\frac{2p_j}{(||\mathbf{p}||^2+e)},\\
&\quad+\frac{1}{2}\sum_{j=1}^{n}\left[[\sigma_{j,j}(l_j+p_j)+\boldsymbol{\Lambda}_{j,j}(t,\mathbf{p})]^2+\sum_{l\neq j}^{n}\boldsymbol{\Lambda}_{j,l}(t,\mathbf{p})^2\right]\Bigg[\frac{2}{(||\mathbf{p}||^2+e)}\\
&\quad-\frac{4p_j^2}{(||\mathbf{p}||^2+e)^2}\Bigg]\\
&\quad+\frac{1}{2}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\left[\sum_{l=1}^{n}\boldsymbol{\Lambda}_{i,l}\boldsymbol{\Lambda}_{j,l}+2[\sigma_{i,i}(l_i+p_i)+\boldsymbol{\Lambda}_{i,i}]\boldsymbol{\Lambda}_{j,i}\right]\left[-\frac{4p_ip_j}{(||\mathbf{p}||^2+e)^2}\right]\\
&= -\sum_{j=1}^{n}\frac{\alpha_j}{2}\left(y_j+l_j\right)^2+\sum_{j=1}^{n}\alpha_j\left(-\left[p_j-\frac{y_j-l_j}{2}\right]^2+\left(\frac{y_j+l_j}{2}\right)^2\right)\frac{2p_j}{(||\mathbf{p}||^2+e)}\\
&\quad+\sum_{j=1}^{n}\frac{2\boldsymbol{h}_j(t,\mathbf{p})p_j}{(||\mathbf{p}||^2+e)}\\
&\quad+\frac{1}{2}\sum_{j=1}^{n}[\sigma_{j,j}(l_j+p_j)+\boldsymbol{\Lambda}_{j,j}(t,\mathbf{p})]^2\left(\frac{2}{(||\mathbf{p}||^2+e)}-\frac{4p_j^2}{(||\mathbf{p}||^2+e)^2}\right)\\
&\quad+\frac{1}{2}\sum_{j=1}^{n}\sum_{l\neq j}^{n}\boldsymbol{\Lambda}_{j,l}(t,\mathbf{p})^2\left(\frac{2}{(||\mathbf{p}||^2+e)}-\frac{4p_j^2}{(||\mathbf{p}||^2+e)^2}\right)\\
&\quad-\frac{1}{2}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\left[\sum_{l=1}^{n}\boldsymbol{\Lambda}_{i,l}\boldsymbol{\Lambda}_{j,l}+2[\sigma_{i,i}(l_i+p_i)+\boldsymbol{\Lambda}_{i,i}]\boldsymbol{\Lambda}_{j,i}\right]\frac{4p_ip_j}{(||\mathbf{p}||^2+e)^2}\\
&\leq -\sum_{j=1}^{n}\frac{\alpha_j}{2}\left(y_j+l_j\right)^2+\sum_{j=1}^{n}\frac{\alpha_j}{2}\left(y_j+l_j\right)^2\frac{p_j}{(||\mathbf{p}||^2+e)}+\sum_{j=1}^{n}\frac{[a'_{1,j}+\gamma_j||\mathbf{p}||]^2+p_j^2}{(||\mathbf{p}||^2+e)}\\
&\quad+\sum_{j=1}^{n}\frac{[\sigma_{j,j}(l_j+p_j)+\boldsymbol{\Lambda}_{j,j}(t,\mathbf{p})]^2}{(||\mathbf{p}||^2+e)}\\
&\quad+\sum_{j=1}^{n}\sum_{l\neq j}^{n}(a'_{2,j}+\tilde{\sigma}_{j,l})^2+2\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{l\neq i,j}^{n}\left(a'_{2,i}+\tilde{\sigma}_{i,l}\right)\left(a'_{2,j}+\tilde{\sigma}_{j,l}\right)\\
&\quad+4\sum_{i=1}^{n}\sum_{j\neq i}^{n}(a'_{2,j}+\tilde{\sigma}_{j,i})\left[\sigma_{i,i}(l_i+1)+(a'_{2,i}+\tilde{\sigma}_{i,i})\right]\\
&\leq cV_2(t,\mathbf{p}),
\end{aligned}
$$

where $c=1+\sum_{j=1}^{n}[a'_{1,j}+\gamma_j]^2+2\sum_{j=1}^{n}\left[\sigma_{j,j}^2(l_j+1)^2+(a'_{2,j}+\tilde{\sigma}_{j,j})^2\right]+\sum_{j=1}^{n}\sum_{l\neq j}^{n}(a'_{2,j}+\tilde{\sigma}_{j,l})^2+2\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sum_{l\neq i,j}^{n}\left(a'_{2,i}+\tilde{\sigma}_{i,l}\right)\left(a'_{2,j}+\tilde{\sigma}_{j,l}\right)+4\sum_{i=1}^{n}\sum_{j\neq i}^{n}(a'_{2,j}+\tilde{\sigma}_{j,i})\left[\sigma_{i,i}(l_i+1)+(a'_{2,i}+\tilde{\sigma}_{i,i})\right]$.

Furthermore, $\inf_{||p||>m}V_{2,m}(t,\mathbf{p})\to\infty$ as $m\to\infty$. $\square$

Following the final argument used in proving the global existence of $\mathbf{y}$ in Lemma 9.1, we conclude that there exists a unique global solution to the interconnected system of stochastic differential (9.12).

In the next section, we discuss about the case where we incorporate jump process into the system $(\mathbf{y}, \mathbf{p})$.

9.4 Energy Commodity Model With and Without Jumps

Due to random interventions that affects the price of energy commodities, we introduce random interventions described by a continuous jump in our model. We follow the approach discussed in [120, 138]. In their work, Wu [120, 138] investigated a class of stochastic hybrid dynamic processes.

Let $K \geq 0$ be the number of jumps on the interval $[t_0, T]$, for $T > 0$. For $K \neq 0$, let $T_1, ..., T_K$ be the jump times over a time interval $[t_0, T]$ such that $t_0 = T_0 \leq T_1 < ... < T_K \leq T$, where T_i denotes the time at which the i-th jump occurred in the system $(\mathbf{y}, \mathbf{p})$. For $K = 0$, no jump has occurred on the interval $[t_0, T]$. We denote the i-th sub-interval by $T_{i-1} \leq t < T_i$. Knowing the global existence and uniqueness solution process of system (9.12) on the interval $[t_0, T]$, $T > 0$ in Section 9.3, for $i \in I(1, K^*)$ and $K \neq 0$, we consider the solution process on each subinterval $[T_{i-1}, T_i)$, between jumps, where $K^* = K$ if $T_K = T$, and $K^* = K + 1$ if $T_K < T$. For $i \in I(1, K^*)$ and $K = 0$, we have $I(1, K) = \emptyset$ or $I(1, K^*) = \{1\}$. In this case, we consider the solution process on $[t_0, T]$. The interconnected system is governed by the following system of continuous time stochastic process;

$$\begin{cases} d\mathbf{y}^{i-1} &= \mathbf{a}^{i-1}(t, \mathbf{y})dt + \mathbf{\Upsilon}^{i-1}(t, \mathbf{y})d\mathbf{W}(t), \quad \mathbf{y}\left(T_{i-1}\right) = \mathbf{y}^{i-1}, \quad t \in [T_{i-1}, T_i) \\ d\mathbf{p}^{i-1} &= \mathbf{b}^{i-1}(t, \mathbf{y}, \mathbf{p})dt + \boldsymbol{\sigma}^{i-1}(t, \mathbf{p})d\mathbf{Z}(t), \quad \mathbf{p}\left(T_{i-1}\right) = \mathbf{p}^{i-1}, \quad t \in [T_{i-1}, T_i),\ i \in I(1, K^*) \\ \mathbf{y}^i &= \mathbf{\Pi}^i \mathbf{y}^{i-1}\left(T_i^-, T_{i-1}, \mathbf{y}^{i-1}\right), \\ \mathbf{p}^i &= \mathbf{\Theta}^i \mathbf{p}^{i-1}\left(T_i^-, T_{i-1}, \mathbf{y}^{i-1}, \mathbf{p}^{i-1}\right), \end{cases} \tag{9.23}$$

where

$$\begin{aligned} \mathbf{\Pi}^i &= \operatorname{diag}\left(\pi_1^i, \pi_2^i, ..., \pi_n^i\right), \\ \mathbf{\Theta}^i &= \operatorname{diag}\left(\theta_1^i, \theta_2^i, ..., \theta_n^i\right), \end{aligned}$$

$\left(\mathbf{y}^{i-1}(t, T_{i-1}, \mathbf{y}^{i-1}), \mathbf{p}^{i-1}(t, T_{i-1}, \mathbf{y}^{i-1}, \mathbf{p}^{i-1})\right)$ is the solution of system of equation (9.12); for $K \neq 0$ and $i \in I(1, K^*)$, $\mathbf{\Pi}^i$ and $\mathbf{\Theta}^i$ are jump coefficient matrices. These jump coefficients are estimated by $\hat{\pi}^i_j = \frac{y_j(T_i)}{\lim\limits_{t\to T_i^-} y_j^{i-1}(t,T_{i-1},\mathbf{y}^{i-1})}$; $\hat{\theta}^i_j = \frac{p_j(T_i)}{\lim\limits_{t\to T_i^-} p_j^{i-1}(t,T_{i-1},\mathbf{y}^{i-1},\mathbf{p}^{i-1})}$ for $j \in I(1,n)$. Following the approach in [120, 138], the solution of (9.23) takes the following representation:

$$\mathbf{y}(t,t_0,\mathbf{y}_0) = \begin{cases} \mathbf{y}^0\left(t,t_0,\mathbf{y}_0\right), & \mathbf{y}(t_0)=\mathbf{y}_0, \quad t_0 \le t < T_1 \\ \mathbf{y}^1\left(t,T_1,\mathbf{y}^1\right), & \mathbf{y}^1 = \mathbf{y}\left(T_1\right), \quad T_1 \le t < T_2, \\ ... \\ \mathbf{y}^{i-1}\left(t,T_{i-1},\mathbf{y}^{i-1}\right), & \mathbf{y}^{i-1} = \mathbf{y}\left(T_{i-1}\right), \quad T_{i-1} \le t < T_i, \\ ... \\ \mathbf{y}^K\left(t,T_K,\mathbf{y}^K\right), & \mathbf{y}^K = \mathbf{y}(T_K), \quad T_K \le t \le T, \end{cases}$$
$$\mathbf{p}(t,t_0,\mathbf{y}_0,\mathbf{p}_0) = \begin{cases} \mathbf{p}^0\left(t,t_0,\mathbf{y}_0,\mathbf{p}_0\right), & \mathbf{p}(t_0)=\mathbf{p}_0, \quad t_0 \le t < T_1 \\ \mathbf{p}^1\left(t,T_1,\mathbf{y}^1,\mathbf{p}^1\right), & \mathbf{p}^1 = \mathbf{p}(T_1), \quad T_1 \le t < T_2 \\ ... \\ \mathbf{p}^{i-1}\left(t,T_{i-1},\mathbf{y}^{i-1},\mathbf{p}^{i-1}\right), & \mathbf{p}^{i-1} = \mathbf{p}(T_{i-1}), \quad T_{i-1} \le t < T_i, \\ ... \\ \mathbf{p}^K\left(t,T_K,\mathbf{y}^K,\mathbf{p}^K\right), & \mathbf{p}^K = \mathbf{p}\left(T_K\right), \quad T_K \le t \le T, \end{cases} \tag{9.24}$$

and $I(1,0) = \{i \in \mathbb{Z} \,:\, 1 \le i \le 0\} = \emptyset$ and $I(1,K^*) = \{1\}$.

REMARK 24 For no jump, that is $K = 0$, (9.23) and (9.24) reduce to

$$\begin{cases} d\mathbf{y} = \mathbf{a}(t,\mathbf{y})dt + \boldsymbol{\Upsilon}(t,\mathbf{y})d\mathbf{W}(t),\ \mathbf{y}(t_0) = \mathbf{y}_0 \\ d\mathbf{p} = \mathbf{b}(t,\mathbf{y},\mathbf{p})dt + \boldsymbol{\sigma}(t,\mathbf{p})d\mathbf{Z}(t),\ \mathbf{p}(t_0) = \mathbf{p}_0,\ t_0 \le t \le T; \end{cases} \tag{9.25}$$

and

$$\begin{cases} \mathbf{y}(t,t_0,\mathbf{y}_0), & \mathbf{y}(t_0) = \mathbf{y}_0, \\ \mathbf{p}(t,t_0,\mathbf{y}_0,\mathbf{p}_0), & \mathbf{p}(t_0) = \mathbf{p}_0,\ t_0 \le t < T, \end{cases} \tag{9.26}$$

respectively.

9.5 Multivariate Discrete Time Dynamic Model for Local Sample Mean and Covariance Process

In this section, we use the idea of moving average to derive an algorithm for the mean and covariance of sample sequences with respect to a continuous time stochastic process. The development of idea and model of statistic for mean and covariance processes is motivated by the state and parameter estimation problems of continuous time nonlinear stochastic dynamic model of the energy commodity market network (4.11) . For this purpose, we need to introduce a few definitions and notations.

For each $i \in I(1, K^*)$, let τ_{i-1} and γ_{i-1}, be finite constant time delays such that $0 < \gamma_{i-1} \leq \tau_{i-1}$. If $K = 0$, then there is no jump. Hence, we simply write $\tau_{i-1} = \tau$ and $\gamma_{i-1} = \gamma$. Here τ_{i-1} characterize the influence of the past performance history of state of dynamic process. γ_{i-1} describe the reaction or response time delays. In general, these time delays are unknown and random variables. These types of delays play an important role in developing mathematical models of continuous time [64] and discrete time [59, 88] dynamic processes. Based upon the practical nature of data collection process, it is essential to either transform these time delays into positive integers or design a suitable data collection schedule or discretization process. For this purpose, for each $i \in I(1, K^*)$, we describe the discrete version of time delays of τ_i and γ_i as follows:

$$\left\{ \begin{array}{lcl} r_{i-1} & = & \left[\left|\frac{\tau_{i-1}}{\Delta t_{i-1}}\right|\right] + 1, \text{ and } \eta_{i-1} = \left[\left|\frac{\gamma_{i-1}}{\Delta t_{i-1}}\right|\right] + 1, \text{ for } i \in I(1, K^*). \end{array} \right. \tag{9.27}$$

Moreover, for the sake of simplicity, we assume that $0 < \gamma_{i-1} < 1, (\eta_{i-1} = 1)$.

DEFINITION 9.5.1 *Let $\boldsymbol{x} = [x_1, x_2, ..., x_n]^T$ be a continuous time multivariate stochastic process defined on an interval $[t_0-\tau, T]$ into $\mathbb{R}^n$, for some $T > 0$. For $t \in [t_0-\tau, T]$, let $\mathcal{F}_t$ be an increasing sub-sigma algebra of a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ for which $\boldsymbol{x}(t)$ is $\mathcal{F}_t$ measurable. For each $i \in I(1, K^*)$, let $\mathbb{P}$ and $\mathbb{P}^{i-1}$ be a partition of $[t_0 - \tau, T]$ and $[T_{i-1} - \tau_{i-1}, T_i]$, respectively. The partition $\mathbb{P}^{i-1}$ is derived by decomposing the partition $\mathbb{P}$. For each $i \in I(1, K^*)$, the partitions $\mathbb{P}$ and $\mathbb{P}^{i-1}$ are defined as follows:*

$$\left\{ \begin{array}{lcl} \mathbb{P} & = & \{t_k \,:\, t_k = t_0 + k\Delta t, \;\; k \in I(-r, N)\}, \\ \mathbb{P}^{i-1} & = & \{t_k^{i-1} \,:\, t_k^{i-1} = T_{i-1} + k\Delta t, \; k \in I(-r_{i-1}, N_{i-1})\}, \end{array} \right. \tag{9.28}$$

where $\Delta t = \frac{T-t_0}{N} = \frac{T_i - T_{i-1}}{N_{i-1}}$ and $T_0 = t_0$.

REMARK 25 We define $\mathcal{S}_i = \sum_{l=0}^{i} N_{l-1}$ with $\mathcal{S}_0 = 0$. For $K \neq 0$, we note that we can write $\mathbb{P}$ as $\{t_0 < t_1 < ... < t_{N_0} < t_{N_0+1} < .... < t_{N_0+N_1} < t_{N_0+N_1+1} < .... < t_{\mathcal{S}_{i-1}} < t_{\mathcal{S}_{i-1}+1} < ... < t_{\mathcal{S}_i} < ... < T\}$. From this, it follows directly that the jump times T_i are contained in $\mathbb{P}$. For any $t_k^{i-1} \in \mathbb{P}^{i-1}$, $k \in [0, N_{i-1}]$, we have $t_k^{i-1} \in \mathbb{P}$. Hence, there is a relationship between elements of $\mathbb{P}^{i-1}$ with $\mathbb{P}$ that is described by $t_k^{i-1} = t_{\mathcal{S}_{i-1}+k}$, for $k \in I(0, N_i)$. In fact, the relationship between set of jump times $\{T_1, T_2, ..., T_K\}$ and the partition $\mathbb{P}$ defined in (9.28) is as: $T_i \to t_{\mathcal{S}_i}$, where the

N_{j-1}'s are the size of partition $\mathbb{P}^{i-1}$ of the sub-interval $[T_{j-1}, T_j]$. It follows that $\mathcal{S}_{K^*} = N$. Using these facts, and noting that if $K = 0$, then $\mathbb{P}^{i-1} = \mathbb{P}$, $N_{i-1} = N$, $\tau_{i-1} = \tau$, $\gamma_{i-1} = \gamma$, $r_{i-1} = r$, $\eta_{i-1} = \eta$, $t_k^{i-1} = t_k$. Moreover, (9.28) can be written as:

$$\mathbb{P}^{i-1} = \left\{ t_k^{i-1} \ : \ t_k^{i-1} = T_{i-1} + k\Delta t, \ k \in I(-r_{i-1}, N_{i-1}) \right\}. \tag{9.29}$$

For each $i \in I(1, K^*)$, let $\{\mathbf{x}^{i-1}(t_k^{i-1})\}_{k=-r_{i-1}}^{N_{i-1}}$ be a finite sequence corresponding to the stochastic process $\mathbf{x}$ and partition $\mathbb{P}^{i-1}$ defined in (9.29). We simply write $\mathbf{x}(t_k^{i-1}) \equiv \mathbf{x}^{i-1}(t_k^{i-1})$. We further recall that $\mathbf{x}(t_k^{i-1})$ is $\mathcal{F}_{t_k^{i-1}}$ measurable for $k \in I(-r_{i-1}, N_{i-1})$. We also recall the definition of forward time shift operator F [11] :

$$F^l \mathbf{x}\left(t_k^{i-1}\right) = \mathbf{x}\left(t_{k+l}^{i-1}\right), \ l \in I(0, \infty). \tag{9.30}$$

DEFINITION 9.5.2 *For $q_{i-1} = 1$ and $r_{i-1} \geq 1$, each $k \in I\left(0, N_{i-1}\right)$, and each $m_k^{i-1} \in I(2, r_{i-1} + \mathcal{S}_{i-1} + k - 1)$, a partition P_k^{i-1} of closed interval $[t_{k-m_k^{i-1}}^{i-1}, t_{k-1}^{i-1}]$ is called local at time t_k^{i-1}, and it is defined by*

$$P_k^{i-1} := t_{k-m_k^{i-1}}^{i-1} < t_{k-m_k^{i-1}+1}^{i-1} < \ldots < t_{k-1}^{i-1}. \tag{9.31}$$

Moreover, P_k^{i-1} is referred as the $m_k^{i-1}-$point sub-partition of the partition $\mathbb{P}^{i-1}$ in (9.29) of the closed sub-interval $[t_{k-m_k^{i-1}}^{i-1}, t_{k-1}^{i-1}]$ of $[-\tau_{i-1}, T_i]$.

REMARK 26 We note that for $K = 0$, that is, there is no jump, we have $P_k^{i-1} = P_k$, $m_k^{i-1} = m_k$, $t_{k-m_k^{i-1}}^{i-1} = t_{k-m_k}^{i-1}$, and $t_{k-1}^{i-1} = t_{k-1}$, where P_k is referred as the m_k-point sub-partition of the partition $\mathbb{P}$ in (9.28) of the closed sub-interval $[t_{k-m_k}, t_k]$ of $[t_0 - \tau, T]$ for $k \in I(0, N)$.

DEFINITION 9.5.3 *For each $i \in I(1, K^*)$, $k \in I(0, N_{i-1})$ and $m_k^{i-1} \in I(2, r_{i-1} + \mathcal{S}_{i-1} + k - 1)$, a local finite sequence at t_k^{i-1} of the size m_k^{i-1} is restriction [2] of $\{\boldsymbol{x}(t_k^{i-1})\}_{k=-r_{i-1}}^{Ni-1}$ to P_k^{i-1} in (9.31). This restriction sequence is defined by*

$$\boldsymbol{S}_{m_k^{i-1},k} \quad := \quad \{F^l \boldsymbol{x}(t_{k-1}^{i-1})\}_{l=-m_k^{i-1}+1}^{0}. \tag{9.32}$$

As m_k^{i-1} varies from 2 to $r_{i-1} + \mathcal{S}_{i-1} + k - 1$, the corresponding respective local sequence $\boldsymbol{S}_{m_k^{i-1},k}$ at t_k^{i-1} varies from $\{\boldsymbol{x}(t_l^{i-1})\}_{l=k-2}^{k-1}$ to $\{\boldsymbol{x}(t_l^{i-1})\}_{l=-r_{i-1}+\mathcal{S}_{i-1}+1}^{k-1}$. As a result of this, the sequence defined in (9.32) is also called a m_k^{i-1}-local moving sequence. Furthermore, the average corresponding to the local sequence $\boldsymbol{S}_{m_k^{i-1},k}$ in (9.32) is defined by

$$\bar{\boldsymbol{S}}_{m_k^{i-1},k} \quad = \quad \frac{1}{m_k^{i-1}} \sum_{l=-m_k^{i-1}+1}^{0} F^l \boldsymbol{x}(t_{k-1}^{i-1}). \tag{9.33}$$

The average/mean defined in (9.33) is also called the m_k^{i-1}-local average/mean.

For $i \in I(1,K^*)$, and $k \in I(0, N_{i-1})$, the m_k^{i-1}-local covariance matrix corresponding to the local sequence $\mathbf{S}_{m_k^{i-1},k}$ in (9.32) is defined by

$$\sum\nolimits_{m_k^{i-1},k} = \begin{pmatrix} s^{1,1}_{m_k^{i-1},k} & s^{1,2}_{m_k^{i-1},k} & s^{1,3}_{m_k^{i-1},k} & \cdots & s^{1,n}_{m_k^{i-1},k} \\ s^{2,1}_{m_k^{i-1},k} & s^{2,2}_{m_k^{i-1},k} & s^{2,3}_{m_k^{i-1},k} & \cdots & s^{2,n}_{m_k^{i-1},k} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ s^{n,1}_{m_k^{i-1},k} & s^{n,2}_{m_k^{i-1},k} & s^{n,3}_{m_k^{i-1},k} & \cdots & s^{n,n}_{m_k^{i-1},k}, \end{pmatrix} \tag{9.34}$$

where $s^{j,l}_{m_k^{i-1},k} \equiv s^{j,l}_{m_k^{i-1},k}(x)$, $j,l \in I(1,n)$ is the local sample covariance statistic between x_j and x_l at t_k^{i-1} described by

$$s^{j,l}_{m_k^{i-1},k} := \begin{cases} \frac{1}{m_k^{i-1}} \sum\limits_{a=-m_k^{i-1}+1}^{0} \left(F^a x_j(t_{k-1}^{i-1}) - \frac{1}{m_k^{i-1}} \sum\limits_{b=-m_k^{i-1}+1}^{0} F^b x_j(t_{k-1}^{i-1}) \right) \times \\ \left(F^a x_l(t_{k-1}^{i-1}) - \frac{1}{m_k^{i-1}} \sum\limits_{b=-m_k^{i-1}+1}^{0} F^b x_l(t_{k-1}^{i-1}) \right), \quad \text{for small } m_k^{i-1} \\ \frac{1}{m_k^{i-1}-1} \sum\limits_{a=-m_k^{i-1}+1}^{0} \left(F^a x_j(t_{k-1}^{i-1}) - \frac{1}{m_k^{i-1}} \sum\limits_{b=-m_k^{i-1}+1}^{0} F^b x_j(t_{k-1}^{i-1}) \right) \times \\ \left(F^a x_l(t_{k-1}^{i-1}) - \frac{1}{m_k^{i-1}} \sum\limits_{b=-m_k^{i-1}+1}^{0} F^b x_l(t_{k-1}^{i-1}) \right), \quad \text{for large } m_k^{i-1} \end{cases} \tag{9.35}$$

In the following, we derive a interconnected discrete-time local conditional sample average/mean and covariance dynamic processes. This fundamental result is motivated by Exercise 5.15 in [14]. Denoting $\mathbf{x}(k) \equiv \mathbf{x}(t_k^{i-1})$ for $i \in I(1,K^*)$ and $k \in I(1, N_{i-1})$, we state and prove the following Lemma.

DEFINITION 9.5.4 *Let $\{\mathbb{E}[x_j(t_k^{i-1})|\mathcal{F}_{t_{k-1}^{i-1}}]\}_{k=-r_{i-1}+1}^{N_{i-1}}$ be a conditional random sample of continuous time stochastic dynamic process with respect to sub-σ algebra $\mathcal{F}_{t_k^{i-1}}$, $t_k^{i-1} \in \mathbb{P}^{i-1}$ in (9.29). The m_k^{i-1}-local conditional moving average and covariance defined in the context of (9.33) and (9.34) are called the m_k-local conditional moving sample average/mean and local conditional moving sample variance, respectively.*

LEMMA 9.3 ***(Multivariate Discrete Time Dynamic Model of Local Sample Mean and Sample Covariance Process).*** *Let $\{\mathbb{E}[x_j(t_k^{i-1})|\mathcal{F}_{t_{k-1}^{i-1}}]\}_{k=-r_{i-1}+1}^{N_{i-1}}$ be a conditional random sample defined in Definition (9.5.4). Let $\bar{S}_{m_k^{i-1},k}$ and $\sum_{m_k^{i-1},k}$ be m_k^{i-1}-local conditional sample average and local* *conditional sample covariance at t_k^{i-1}. Then, an interconnected multivariate discrete time dynamic model of local conditional sample mean and sample covariance statistics is described by*

$$\begin{aligned}
\bar{\boldsymbol{S}}_{m^{i-1}_{k-d_{i-1}+1},k-d_{i-1}+1} &= \frac{m^{i-1}_{k-d_{i-1}}}{m^{i-1}_{k-d_{i-1}+1}}\bar{\boldsymbol{S}}_{m^{i-1}_{k-d_{i-1}},k-d_{i-1}} + \boldsymbol{\eta}_{m^{i-1}_{k-d_{i-1}},k-d_{i-1}},\ \ \bar{\boldsymbol{S}}_{m^{i-1}_{T_{i-1}},T_{i-1}} = \bar{\boldsymbol{S}}_{T_{i-1}} \\
\sum\nolimits_{m^{i-1}_k,k} &= \begin{cases}
\frac{m^{i-1}_k-1}{m^{i-1}_k}\left[\sum\limits_{j=1}^{d_{i-1}}\left[\frac{m^{i-1}_{k-j}}{\prod\limits_{l=0}^{j-1} m^{i-1}_{k-l}}\right]\sum\nolimits_{m^{i-1}_{k-j},k-j}\right. \\
\left.+\frac{m^{i-1}_{k-d_{i-1}}}{\prod\limits_{l=0}^{d_{i-1}-1} m^{i-1}_{k-l}}\bar{\boldsymbol{S}}_{m^{i-1}_{k-d_{i-1}},k-d_{i-1}}\bar{\boldsymbol{S}}^T_{m^{i-1}_{k-d_{i-1}},k-d_{i-1}}\right] + \boldsymbol{\varepsilon}_{m^{i-1}_{k-1},k-1}, \\
\textit{for small } m^{i-1}_k, m^{i-1}_{k-1} \le m^{i-1}_k \\
\sum\limits_{j=1}^{d_{i-1}}\left[\frac{m^{i-1}_{k-j}-1}{\prod\limits_{l=0}^{j-1} m^{i-1}_{k-l}}\right]\sum\nolimits_{m^{i-1}_{k-j},k-j} \\
+\frac{m^{i-1}_{k-d_{i-1}}}{\prod\limits_{l=0}^{d_{i-1}-1} m^{i-1}_{k-l}}\bar{\boldsymbol{S}}_{m^{i-1}_{k-d_{i-1}},k-d_{i-1}}\bar{\boldsymbol{S}}^T_{m^{i-1}_{k-d_{i-1}},k-d_{i-1}} \\
+\boldsymbol{\epsilon}_{m^{i-1}_{k-1},k-1},\ \textit{for large } m^{i-1}_k,\ m^{i-1}_{k-1} \le m^{i-1}_k \\
\sum\nolimits_{m^{i-1}_j,j} = \sum\nolimits_j,\ i \in I(1,K^*),\ j \in I(-d_{i-1},0),\ \ \textit{initial conditions}
\end{cases}
\end{aligned} \tag{9.36}$$

where

$$\begin{aligned}
\boldsymbol{\eta} &= \begin{pmatrix} \eta^1 \\ \eta^2 \\ \vdots \\ \eta^n \end{pmatrix}, \\
\boldsymbol{\varepsilon}_{m^{i-1}_k,k} &= \begin{pmatrix}
\varepsilon^{1,1}_{m^{i-1}_k,k} & \varepsilon^{1,2}_{m^{i-1}_k,k} & \varepsilon^{1,3}_{m^{i-1}_k,k} & \cdots & \varepsilon^{1,n}_{m^{i-1}_k,k} \\
\varepsilon^{2,1}_{m^{i-1}_k,k} & \varepsilon^{2,2}_{m^{i-1}_k,k} & \varepsilon^{2,3}_{m^{i-1}_k,k} & \cdots & \varepsilon^{2,n}_{m^{i-1}_k,k} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\varepsilon^{n,1}_{m^{i-1}_k,k} & \varepsilon^{n,2}_{m^{i-1}_k,k} & \varepsilon^{n,3}_{m^{i-1}_k,k} & \cdots & \varepsilon^{n,n}_{m^{i-1}_k,k}
\end{pmatrix}, \\
\boldsymbol{\epsilon}_{m^{i-1}_k,k} &= \begin{pmatrix}
\epsilon^{1,1}_{m^{i-1}_k,k} & \epsilon^{1,2}_{m^{i-1}_k,k} & \epsilon^{1,3}_{m^{i-1}_k,k} & \cdots & \epsilon^{1,n}_{m^{i-1}_k,k} \\
\epsilon^{2,1}_{m^{i-1}_k,k} & \epsilon^{2,2}_{m^{i-1}_k,k} & \epsilon^{2,3}_{m^{i-1}_k,k} & \cdots & \epsilon^{2,n}_{m^{i-1}_k,k} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\epsilon^{n,1}_{m^{i-1}_k,k} & \epsilon^{n,2}_{m^{i-1}_k,k} & \epsilon^{n,3}_{m^{i-1}_k,k} & \cdots & \epsilon^{n,n}_{m^{i-1}_k,k}
\end{pmatrix},
\end{aligned}$$

$$
\begin{aligned}
\eta^{j}_{m^{i-1}_{k-d_{i-1}},k-d_{i-1}} &= \tfrac{1}{m^{i-1}_{k-d_{i-1}+1}}\left[\sum_{\iota=-m^{i-1}_{k-d_{i-1}+1}+1}^{-m^{i-1}_{k-d_{i-1}}+1} F^{\iota}x_j(k-d_{i-1})\right. \\
&\quad -F^{-m^{i-1}_{k-d_{i-1}}+1}x_j(k-d_{i-1}) \\
&\quad \left.-F^{-m^{i-1}_{k-d_{i-1}}}x_j(k-d_{i-1})+F^{0}x_j(k-d_{i-1})\right],
\end{aligned} \tag{9.37}
$$

$$
\begin{aligned}
\varepsilon^{j,l}_{m^{i-1}_{k-1},k-1} &= \frac{m^{i-1}_k-1}{m^{i-1}_k}\left[\sum_{\iota=1}^{d_{i-1}}\frac{F^{-\iota+1}x_j(k-1)F^{-\iota+1}x_l(k-1)}{\prod_{a=0}^{\iota-1}m^{i-1}_{k-a}}\right. \\
&\quad -\sum_{\iota=1}^{d_{i-1}}\frac{F^{-\iota+1-m^{i-1}_{k-\iota}}x_j(k-1)F^{-\iota+1-m^{i-1}_{k-\iota}}x_l(k-1)}{\prod_{a=0}^{\iota-1}m^{i-1}_{k-a}} \\
&\quad \left.-\sum_{\iota=1}^{d_{i-1}}\frac{F^{-\iota+2-m^{i-1}_{k-\iota}}x_j(k-1)F^{-\iota+2-m^{i-1}_{k-\iota}}x_l(k-1)}{\prod_{a=0}^{\iota-1}m^{i-1}_{k-a}}\right] \\
&\quad +\frac{m^{i-1}_k-1}{m^{i-1}_k}\left[\sum_{\iota=1}^{d_{i-1}}\left[\frac{\sum_{v=-\iota+2-m^{i-1}_{k-\iota+1}}^{-\iota+2-m^{i-1}_{k-\iota}} F^{v}x_j(k-1)F^{v}x_l(k-1)}{\prod_{a=0}^{\iota-1}m^{i-1}_{k-a}}\right]\right. \\
&\quad \left.+\sum_{\iota=1}^{d_{i-1}}\left[\frac{\sum_{\substack{v,s=-\iota+2-m^{i-1}_{k-\iota+1}\\ v\neq s}}^{-\iota+1} F^{v}x_j(k-1)F^{s}x_l(k-1)}{\prod_{a=0}^{\iota-1}m^{i-1}_{k-a}}\right]\right] \\
&\quad -\frac{1}{m^{i-1}_k}\sum_{\substack{v,s=-m^{i-1}_k+1\\ v\neq s}}^{0} F^{v}x_j(k-1)F^{s}x_l(k-1),
\end{aligned}
$$

$$
\begin{aligned}
\epsilon^{j,l}_{m^{i-1}_{k-1},k-1} \;=\; & \sum_{\iota=1}^{d_{i-1}} \frac{F^{-\iota+1}x_j(k-1)F^{-\iota+1}x_l(k-1)}{\prod_{a=0}^{\iota-1} m^{i-1}_{k-a}} \\
& -\sum_{\iota=1}^{d_{i-1}} \frac{F^{-\iota+1-m^{i-1}_{k-\iota}}x_j(k-1)F^{-\iota+1-m^{i-1}_{k-\iota}}x_l(k-1)}{\prod_{a=0}^{\iota-1} m^{i-1}_{k-a}} \\
& -\sum_{\iota=1}^{d_{i-1}} \frac{F^{-\iota+2-m^{i-1}_{k-\iota}}x_j(k-1)F^{-\iota+2-m^{i-1}_{k-\iota}}x_l(k-1)}{\prod_{a=0}^{\iota-1} m^{i-1}_{k-a}} \\
& -\frac{1}{m^{i-1}_k-1} \sum_{\substack{v,s=-m^{i-1}_k+1 \\ v\neq s}}^{0} F^v x_j(k-1)F^s x_l(k-1) \\
& +\sum_{\iota=1}^{d_{i-1}} \left[\frac{\sum_{v=-\iota+2-m^{i-1}_{k-\iota+1}}^{-\iota+2-m^{i-1}_{k-\iota}} F^v x_j(k-1)F^v x_l(k-1)}{\prod_{a=0}^{\iota-1} m^{i-1}_{k-a}} \right] \\
& +\sum_{\iota=1}^{d_{i-1}} \left[\frac{\sum_{\substack{v,s=-\iota+2-m^{i-1}_{k-\iota+1} \\ v\neq s}}^{-\iota+1} F^v x_j(k-1)F^s x_l(k-1)}{\prod_{a=0}^{\iota-1} m^{i-1}_{k-a}} \right].
\end{aligned}
$$

9.6 Parametric Estimation

In this section, we consider a parameter estimation problem in drift and diffusion coefficients of (9.23). This is achieved by utilizing the lagged adaptive process [88] and the interconnected discrete-time dynamics of local sample mean and variances statistic processes model in Section 9.5 (Lemma 9.3). For each $i \in I(1, K^*)$, we consider a general interconnected hybrid system described by the system of stochastic differential equations:

$$
\begin{cases}
d\mathbf{x}^{i-1} = \mathbf{f}^{i-1}(t,\mathbf{x})dt + \boldsymbol{\sigma}^{i-1}(t,\mathbf{x})d\mathbf{W}(t), \;\; \mathbf{x}(T_{i-1}) = \mathbf{x}^{i-1}, \;\; t \in [T_{i-1}, T_i), \\
\mathbf{x}^i = \Gamma^i \mathbf{x}^{i-1}(T_i^-, T_{i-1}, \mathbf{x}^{i-1}),
\end{cases}
\tag{9.38}
$$

where $\Gamma^i = \text{diag}(\Gamma^i_1, \Gamma^i_2, ..., \Gamma^i_j, ..., \Gamma^i_n)$ is the jump coefficient matrix; the jump times T_i's are defined in (9.23). For each $j \in I(1, n)$, the estimate of the jump coefficient Γ^i_j is given by $\Gamma^i_j = \frac{x_j(T_i)}{\lim\limits_{t \to T_i^-} x^{i-1}_j(t, T_{i-1}, \mathbf{x}^{i-1})}$.

Let $V \in C[[-\tau, \infty] \times \mathbb{R}^n, \mathbb{R}^m]$, and its partial derivatives V_t, $\frac{\partial V}{\partial x}$ and $\frac{\partial^2 V^{i-1}}{\partial x^2}$ exist and are continuous on each interval $[T_{i-1}, T_i]$. We apply Itô-Doob stochastic differential formula [70] to V, and we obtain

$$\begin{cases} dV(t, \mathbf{x}^{i-1}) &= \mathbf{L}V(t, \mathbf{x}^{i-1})dt + V_{\mathbf{x}}(t, \mathbf{x}^{i-1})\sigma(t, \mathbf{x}^{i-1})dW(t), \ \mathbf{x}(T_{i-1}) = \mathbf{x}^{i-1}, \ \ t \in [T_{i-1}, T_i), \\ V(T_i, \mathbf{x}^i) &= V(T_i, \Gamma^i \mathbf{x}^{i-1}(T_i^-, T_{i-1}, \mathbf{x}^{i-1})). \end{cases} \tag{9.39}$$

where the $\mathbf{L}$ operator is defined by

$$\begin{cases} \mathbf{L}V(t, \mathbf{x}^{i-1}) &= V_t(t, \mathbf{x}^{i-1}) + V_{\mathbf{x}}(t, \mathbf{x}^{i-1})f(t, \mathbf{x}^{i-1}) + \frac{1}{2}tr(V_{\mathbf{xx}}(t, \mathbf{x}^{i-1}))c(t, \mathbf{x}^{i-1}) \\ c(t, \mathbf{x}^{i-1}) &= \boldsymbol{\sigma}^{i-1}(t, \mathbf{x}^{i-1})\boldsymbol{\sigma}^{i-1}(t, \mathbf{x}^{i-1})^T. \end{cases} \tag{9.40}$$

For (9.38) and (9.39), we present the Euler-type discretization scheme [58]:

$$\begin{cases} \Delta \mathbf{x}^{i-1}(t_k^{i-1}) &= \mathbf{f}(t_{k-1}^{i-1}, \mathbf{x}(t_{k-1}^{i-1}))\Delta t_k^{i-1} \\ &\quad +\boldsymbol{\sigma}^{i-1}(t_{k-1}^{i-1}, \mathbf{x}(t_{k-1}^{i-1}))\Delta \mathbf{W}(t_k^{i-1}), \ \ k \in I(1, N_{i-1}) \\ \mathbf{x}^i &= \Gamma^i \mathbf{x}^{i-1}(T_i^-, T_{i-1}, \mathbf{x}^{i-1}), \\ \Delta V(t_k^{i-1}, \mathbf{x}^{i-1}(t_k^{i-1})) &= LV(t_{k-1}^{i-1}, \mathbf{x}^{i-1}(t_{k-1}^{i-1}))\Delta t_k^{i-1} \\ &\quad +V_{\mathbf{x}}(t_{k-1}^{i-1}, \mathbf{x}^{i-1}(t_{k-1}^{i-1}))\boldsymbol{\sigma}^{i-1}(t_{k-1}^{i-1}, \mathbf{x}(t_{k-1}^{i-1}))\Delta W(t_k^{i-1}) \\ V(T_i, \mathbf{x}^i) &= V(T_i, \Gamma^i \mathbf{x}^{i-1}(T_i^-, T_{i-1}, \mathbf{x}^{i-1})). \end{cases} \tag{9.41}$$

Define $\mathcal{F}_{t_{k-1}}^{i-1} \equiv \mathcal{F}_{k-1}^{i-1}$ as the filtration process up to time t_{k-1}^{i-1}. With regard to the continuous time dynamic system (9.38) and its transformed system (9.39), the more general moments of $\Delta \mathbf{x}(t_k^{i-1})$ are as follows:

$$\begin{cases} E\left[\Delta \mathbf{x}(t_k^{i-1})|\mathcal{F}_{k-1}^{i-1}\right] &= \mathbf{f}^{i-1}(t_{k-1}^{i-1}, \mathbf{x}^{i-1}(t_{k-1}^{i-1}))\Delta t_k^{i-1}, \\ E\left[\left(\Delta \mathbf{x}^{i-1}(t_k^{i-1}) - E\left[\Delta \mathbf{x}^{i-1}(t_k^{i-1})|\mathcal{F}_{k-1}^{i-1}\right]\right)\right. & \\ \left.\left(\Delta \mathbf{x}^{i-1}(t_k^{i-1}) - E\left[\Delta \mathbf{x}^{i-1}(t_k^{i-1})|\mathcal{F}_{k-1}^{i-1}\right]\right)^T|\mathcal{F}_{k-1}^{i-1}\right] &= \boldsymbol{\sigma}^{i-1}(t_{k-1}^{i-1}, \mathbf{x}^{i-1}(t_{k-1}^{i-1}))\times \\ &\quad \boldsymbol{\sigma}^{i-1}(t_{k-1}^{i-1}, \mathbf{x}^{i-1}(t_{k-1}^{i-1}))^T\Delta t_k^{i-1}, \\ E\left[\Delta V(t_k^{i-1}, \mathbf{x}^{i-1}(t_k^{i-1}))|\mathcal{F}_{k-1}^{i-1}\right] &= LV(t_{k-1}^{i-1}, \mathbf{x}^{i-1}(t_k^{i-1}),)\Delta t_k^{i-1} \end{cases} \tag{9.42}$$

$$\begin{cases} E\left[\left(\Delta V(t_k^{i-1}, \mathbf{x}^{i-1}(t_k^{i-1})) - E\left[\Delta V(t_k^{i-1}, \mathbf{x}^{i-1}(t_k^{i-1}))|\mathcal{F}_{k-1}^{i-1}\right]\right)\right. \\ \left.\left(\Delta V(t_k^{i-1}, \mathbf{x}^{i-1}(t_k^{i-1})) - E\left[\Delta V(t_k^{i-1}, \mathbf{x}^{i-1}(t_k^{i-1}))|\mathcal{F}_{k-1}^{i-1}\right]\right)^T|\mathcal{F}_{k-1}^{i-1}\right] \ = \ B(t_{k-1}^{i-1}, \mathbf{x}^{i-1}(t_{k-1}^{i-1})) \end{cases} \tag{9.43}$$

where $B(t_{k-1}^{i-1}, \mathbf{x}(t_{k-1}^{i-1})) = V_{\mathbf{x}}\left(t_{k-1}^{i-1}, \mathbf{x}^{i-1}(t_{k-1}^{i-1})\right) c\left(t_{k-1}^{i-1}, \mathbf{x}(t_{k-1}^{i-1})\right) V_{\mathbf{x}}\left(t_{k-1}^{i-1}, \mathbf{x}(t_{k-1}^{i-1})\right)^T \Delta t_k^{i-1}$, and T stands for the transpose of the matrix.

From (9.41)- (9.43), we have

$$\begin{cases} \Delta \mathbf{x}^{i-1}(t_k^{i-1}) & = & E\left[\Delta \mathbf{x}^{i-1}(t_k^{i-1})|\mathcal{F}_{k-1}^{i-1}\right] \\ & & +\boldsymbol{\sigma}^{i-1}(t_{k-1}^{i-1}, \mathbf{x}^{i-1}(t_{k-1}^{i-1}))\Delta \mathbf{W}(t_k^{i-1}), \;\; k \in I(1, N_{i-1}) \\ \Delta V(t_k^{i-1}, \mathbf{x}^{i-1}(t_k^{i-1})) & = & E\left[\Delta V(t_k^{i-1}, \mathbf{x}^{i-1}(t_k^{i-1}))|\mathcal{F}_{k-1}^{i-1}\right] \\ & & +V_{\mathbf{x}}^{i-1}(t_{k-1}^{i-1}, \mathbf{x}^{i-1}(t_{k-1}^{i-1}))\boldsymbol{\sigma}^{i-1}(t_{k-1}^{i-1}, \mathbf{x}^{i-1}(t_{k-1}^{i-1}))\Delta W(t_k) \end{cases} \tag{9.44}$$

This provides the basis for the development of the concept of lagged adaptive expectation process [88] with respect to continuous time stochastic dynamic system (9.39). This indeed leads to a formulation of m_k^{i-1}-local generalized method of moments at t_k^{i-1}.

In the following, we state a result that exhibits the existence of solution of system of non linear equations. For the sake of easy reference, we shall re-state the Implicit function theorem without proof.

THEOREM 9.1 ***Implicit Function Theorem[2]*** *Let $\boldsymbol{F} = \{F_1, F_2, ..., F_q\}$ be a vector-valued function defined on an open set $S \in \mathbb{R}^{q+k}$ with values in $\mathbb{R}^q$. Suppose $\boldsymbol{F} \in \mathcal{C}_1$ on S. Let $(\boldsymbol{u}_0; \boldsymbol{v}_0)$ be a point in S for which $\boldsymbol{F}(\boldsymbol{u}_0; \boldsymbol{v}_0) = 0$ and for which the $q \times q$ determinant of the Jacobian matrix det $[J_{\boldsymbol{F}}(\boldsymbol{v}_0)] \neq 0$. Then there exists a $k-$ dimensional open set $\boldsymbol{T}_0$ containing $\boldsymbol{v}_0$ and unique vector-valued function $\boldsymbol{g}$, defined on $\boldsymbol{T}_0$ and having values in $\mathbb{R}^q$, such that $\boldsymbol{g} \in \mathcal{C}_1$ on $\boldsymbol{T}_0$, $\boldsymbol{g}(\boldsymbol{v}_0) = \boldsymbol{u}_0$, and $\boldsymbol{F}(\boldsymbol{g}(\boldsymbol{v}); \boldsymbol{v}) = 0$ for every $\boldsymbol{v} \in \boldsymbol{T}_0$.*

9.6.1 Illustration:

For each $j, l \in I(1, n)$ and each $i \in I(1, K^*)$, we consider a special case of (9.12).

$$\begin{cases} dy_j & = & \left(u_j^{i-1} - y_j\right)\left[\kappa_{j,j}^{i-1} y_j + \sum_{l \neq j}^{n} \kappa_{j,l}^{i-1} y_l\right] dt + \delta_{j,j}^{i-1}\left(u_j^{i-1} - y_j\right) dW_{j,j}(t) \\ & & +\left(u_j^{i-1} - y_j\right)\sum_{l \neq j}^{n} \delta_{j,l}^{i-1} y_l dW_{j,l}(t), \; y_j\left(T_{i-1}\right) = y_j^{i-1}, \; t \in [T_{i-1}, T_i), \\ y_j^i & = & \pi_j^i y_j^{i-1}\left(T_i^-, T_{i-1}, \mathbf{y}^{i-1}\right), \\ dp_j(t) & = & p_j\left[\gamma_{j,j}^{i-1}\left(y_j - p_j\right) + \beta_j^{i-1} + \sum_{l \neq j}^{n} \gamma_{j,l}^{i-1} p_l(t)\right] dt + \sigma_{j,j}^{i-1} p_j dZ_{j,j}(t) \\ & & +p_j \sum_{l \neq j}^{n} \sigma_{j,l}^{i-1} p_l dZ_{j,l}(t), \; p_j(T_{i-1}) = p_j^{i-1}, \; t \in [T_{i-1}, T_i), \\ p_j^i & = & \theta_j^i p_j^{i-1}\left(T_i^-, T_{i-1}, \mathbf{y}^{i-1}, \mathbf{p}^{i-1}\right). \end{cases} \tag{9.45}$$

Here, $\kappa_{j,l}^{i-1}$, u_j^{i-1}, β_j^{i-1}, $\gamma_{j,l}^{i-1}$, $\delta_{j,l}^{i-1}$, $\sigma_{j,l}^{i-1}$ are the system parameters on the jump subinterval $[T_{i-1}, T_i)$; u_j^{i-1}, $\kappa_{j,j}^{i-1}$, $\gamma_{j,j}^{i-1}$, $\delta_{j,j}^{i-1}$ and $\sigma_{j,j}^{i-1}$ are positive; and for $l \neq j$, $\kappa_{j,l}^{i-1}, \gamma_{j,l}^{i-1} \in \mathbb{R}$; $\delta_{j,l}^{i-1}$, $\sigma_{j,l}^{i-1}$ are nonnegative. W and Z are independent standard Wiener process on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{t\geq 0}, \mathcal{P})$ with the properties described in (9.12). It follows that the interconnected system of stochastic differential equations (9.45) has $4n^2 + 2n$ parameters, Also,

$$\{\kappa_{j,l}\}_{l\neq j}^{i-1} \begin{cases} > 0 & \text{if } y_l \text{ is cooperating with } y_j, \\ < 0 & \text{if } y_l \text{ is competing with } y_j, \\ = 0 & \text{if there is no interaction between } y_l \text{ and } y_j, \quad j,l \in I(1,n), \end{cases} \tag{9.46}$$

and likewise,

$$\{\gamma_{j,l}\}_{l\neq j}^{i-1} \begin{cases} > 0 & \text{if } p_l \text{ is cooperating with } p_j, \\ < 0 & \text{if } p_l \text{ is competing with } p_j, \\ = 0 & \text{if there is no interaction between } p_l \text{ and } p_j, \quad j,l \in I(1,n). \end{cases} \tag{9.47}$$

REMARK 27 For the case $K = 0$, (9.45) reduce to

$$\begin{cases} dy_j &= (u_j - y_j)\left[\kappa_{j,j} y_j + \sum\limits_{l\neq j}^{n} \kappa_{j,l} y_l\right] dt + \delta_{j,j}\,(u_j - y_j)\, dW_{j,j}(t) \\ & \quad + (u_j - y_j) \sum\limits_{l\neq j}^{n} \delta_{j,l} y_l dW_{j,l}(t),\ y_j\,(t_0) = y_{j0},\ t \in [t_0, T], \\ dp_j(t) &= p_j\left[\gamma_{j,j}\,(y_j - p_j) + \beta_j + \sum\limits_{l\neq j}^{n} \gamma_{j,l} p_l(t)\right] dt + \sigma_{j,j} p_j dZ_{j,j}(t) \\ & \quad + p_j \sum\limits_{l\neq j}^{n} \sigma_{j,l} p_l dZ_{j,l}(t),\ p_j\,(t_0) = p_{j0},\ t \in [t_0, T], \end{cases} \tag{9.48}$$

where for $j, l \in I(1,n)$, the parameters $\kappa_{j,l}$, u_j, β_j, $\gamma_{j,l}$, $\delta_{j,l}$ and $\sigma_{j,l}$ are the system parameters on the interval $[t_0, T]$; u_j, $\kappa_{j,j}$, $\gamma_{j,j}$, $\delta_{j,j}$ and $\sigma_{j,j}$ are positive; and for $l \neq j$, $\kappa_{j,l}, \gamma_{j,l} \in \mathbb{R}$; $\delta_{j,l}$, $\sigma_{j,l}$ are nonnegative.

For each $j \in I(1,n)$, we pick a Lyapunov function

$$\begin{cases} V_{1j}(t, y_j) &= (y_j)^q, \\ V_{2j}(t, p_j) &= (p_j)^q, \end{cases} \tag{9.49}$$

in (9.39) for (9.45). Using Itô-differential formula [70], we have

$$
\begin{cases}
dV_{1j} &= \left[q\left(y_j\right)^{q-1}\left(u_j^{i-1}-y_j\right)\left(\kappa_{j,j}^{i-1}y_j+\sum\limits_{l\neq j}^{n}\kappa_{j,l}^{i-1}y_l\right)\right. \\
& \quad +\frac{1}{2}q(q-1)\left(y_j\right)^{q-2}\left(u_j^{i-1}-y_j\right)^2\left.\left(\left(\delta_{j,j}^{i-1}\right)^2+\sum\limits_{l\neq j}^{n}\left(\delta_{j,l}^{i-1}\right)^2 y_l^2\right)\right]dt \\
& \quad +q\left(y_j\right)^{q-1}\left(u_j^{i-1}-y_j\right)\left[\delta_{j,j}^{i-1}dW_{j,j}(t)+\sum\limits_{l\neq j}^{n}\delta_{j,l}^{i-1}y_l dW_{j,l}(t)\right], \\
& \quad y_j\left(T_{i-1}\right)=y_j^{i-1},\ t\in[T_{i-1},T_i) \\
V_{1j}^i &= \left(\pi_j^i\right)^q y_j\left(T_i^-,T_{i-1},\mathbf{y}^{i-1}\right)^q,\ \text{if } t=T_i, \\
dV_{2j} &= \left(p_j\right)^q\left[q\left(\gamma_j^{i-1}(y_j-p_j)+\beta_j^{i-1}+\sum\limits_{l\neq j}^{n}\gamma_{j,l}^{i-1}p_l\right)\right. \\
& \quad +\frac{1}{2}q(q-1)\left.\left(\left(\sigma_{j,j}^{i-1}\right)^2+\sum\limits_{l\neq j}^{n}\left(\sigma_{j,l}^{i-1}\right)^2 p_l^2\right)\right]dt \\
& \quad +q\left(p_j\right)^q\left[\sigma_{j,j}^{i-1}dZ_{j,j}(t)+\sum\limits_{l\neq j}^{n}\sigma_{j,l}^{i-1}p_l dZ_{j,l}(t)\right],\ p_j\left(T_{i-1}\right)=p_j^{i-1},\ t\in[T_{i-1},T_i), \\
V_{2j}^i &= \left(\theta_j^i\right)^q p_j\left(T_i^-,T_{i-1},\mathbf{y}^{i-1},\mathbf{p}^{i-1}\right)^q,\ \text{if } t=T_i,
\end{cases}
\tag{9.50}
$$

By setting $\Delta t_k^{i-1}=t_k^{i-1}-t_{k-1}^{i-1}=\Delta t$; $\Delta\mathbf{y}(t_k^{i-1})=\mathbf{y}(t_k^{i-1})-\mathbf{y}(t_{k-1}^{i-1})$ and $\Delta\mathbf{p}(t_k^{i-1})=\mathbf{p}(t_k^{i-1})-\mathbf{p}(t_{k-1}^{i-1})$, the combined Euler discretized scheme for (9.50) is

$$
\begin{cases}
\Delta\left(y_j\right)^q\left(t_k^{i-1}\right) &= \left[q\left(y_j\right)^{q-1}\left(t_{k-1}^{i-1}\right)\left(u_j^{i-1}-y_j(t_{k-1}^{i-1})\right)\left(\kappa_{j,j}^{i-1}y_j(t_{k-1}^{i-1})+\sum\limits_{l\neq j}^{n}\kappa_{j,l}^{i-1}y_l(t_{k-1}^{i-1})\right)\right. \\
& \quad +\frac{1}{2}q(q-1)\left(y_j\right)^{q-2}\left(t_{k-1}^{i-1}\right)\left(u_j^{i-1}-y_j(t_{k-1}^{i-1})\right)^2\left(\left(\delta_{j,j}^{i-1}\right)^2\right. \\
& \quad \left.\left.+\sum\limits_{l\neq j}^{n}\left(\delta_{j,l}^{i-1}\right)^2 y_l^2(t_{k-1}^{i-1})\right)\right]\Delta t \\
& \quad +q\left(y_j\right)^{q-1}\left(t_{k-1}^{i-1}\right)\left(u_j^{i-1}-y_j(t_{k-1}^{i-1})\right)\left[\delta_{j,j}^{i-1}\Delta W_{j,j}(t_k^{i-1})\right. \\
& \quad \left.+\sum\limits_{l\neq j}^{n}\delta_{j,l}^{i-1}y_l\Delta W_{j,l}(t_k^{i-1})\right], \\
& \quad y_j\left(T_{i-1}\right)=y_j^{i-1},\ \ t_k^{i-1}\in[T_{i-1},T_i), \\
\left(y_j^i\right)^q &= \left(\pi_j^i\right)^q y_j^{i-1}\left(T_i^-,T_{i-1},\mathbf{y}^{i-1}\right)^q,\ \text{if } t=T_i,
\end{cases}
\tag{9.51}
$$

$$\begin{cases} \Delta\left(p_j\right)^q\left(t_k^{i-1}\right) &= \left(p_j\right)^q\left(t_{k-1}^{i-1}\right)\left[q\left(\gamma_{j,j}^{i-1}(y_j(t_{k-1}^{i-1})-p_j(t_{k-1}^{i-1}))+\beta_j^{i-1}+\sum\limits_{l\neq j}^{n}\gamma_{j,l}^{i-1}p_l(t_{k-1}^{i-1})\right)\right. \\ & \left. +\frac{1}{2}q(q-1)\left(\left(\sigma_{j,j}^{i-1}\right)^2+\sum\limits_{l\neq j}^{n}\left(\sigma_{j,l}^{i-1}\right)^2 p_l^2(t_{k-1}^{i-1})\right)\right]\Delta t \\ & +q\left(p_j\right)^q\left(t_{k-1}^{i-1}\right)\left[\sigma_{j,j}^{i-1}\Delta Z_{j,j}(t_k^{i-1})+\sum\limits_{l\neq j}^{n}\sigma_{j,l}^{i-1}p_l(t_{k-1}^{i-1})\Delta Z_{j,l}(t_k^{i-1})\right], \\ & p_j\left(T_{i-1}\right)=p_j^{i-1}\; t_k^{i-1}\in[T_{i-1},T_i),\; q\in I(1,n+1), \\ \left(p_j^i\right)^q &= \left(\theta_j^i\right)^q p_j^{i-1}\left(T_i^-,T_{i-1},\mathbf{y}^{i-1},\mathbf{p}^{i-1}\right)^q,\ \text{if } t=T_i. \end{cases} \tag{9.52}$$

where $\{\mathbf{y}(t_k^{i-1})\}_{k=-r_{i-1}}^{0}$, $\{\mathbf{p}(t_k^{i-1})\}_{k=-r_{i-1}}^{0}$ are given finite sequence of $\mathcal{F}_{T_{i-1}}^{i-1}-$ measurable random vectors, and are independent of $\{\Delta W(t_k^{i-1})\}_{k=0}^{N_{i-1}}$, $\{\Delta Z(t_k^{i-1})\}_{k=0}^{N_{i-1}}$, respectively. We define $\Delta\left(y_j\right)^q\left(t_k^{i-1}\right)=\left(y_j\right)^q\left(t_k^{i-1}\right)-\left(y_j\right)^q\left(t_{k-1}^{i-1}\right)$ and $\Delta\left(p_j\right)^q\left(t_k^{i-1}\right)=\left(p_j\right)^q\left(t_k^{i-1}\right)-\left(p_j\right)^q\left(t_{k-1}^{i-1}\right)$.

Applying conditional expectations to (9.51)-(9.52) with respect to $\mathcal{F}_{t_{k-1}}^{i-1}\equiv\mathcal{F}_{k-1}^{i-1}$, we obtain

$$\begin{cases} \mathbb{E}\left[\Delta\left(y_j\right)^q\left(t_k^{i-1}\right)|\mathcal{F}_{k-1}^{i-1}\right]= \\ \left[q\left(y_j\right)^{q-1}\left(t_{k-1}^{i-1}\right)\left(u_j^{i-1}-y_j(t_{k-1}^{i-1})\right)\left(\kappa_{j,j}^{i-1}y_j(t_{k-1}^{i-1})+\sum\limits_{l\neq j}^{n}\kappa_{j,l}^{i-1}y_l(t_{k-1}^{i-1})\right)\right. \\ \left.+\frac{q(q-1)}{2\Delta t_i}\left(y_j\right)^{q-2}\left(t_{k-1}^{i-1}\right)\mathbb{E}\left[\left(\Delta y_j(t_k^{i-1})-\mathbb{E}[\Delta y_j(t_k^{i-1})|\mathcal{F}_{k-1}^{i-1}]\right)^2|\mathcal{F}_{k-1}^{i-1}\right]\right]\Delta t \\ \text{for } t_k^{i-1}\in[T_{i-1},T_i), \\ \mathbb{E}\left[\left(y_j^i\right)^q\left(t_k^{i-1}\right)|\mathcal{F}_{k-1}^{i-1}\right]=\left(\pi_j^i\right)^q\left(y_j\right)^q\left(T_i^-,T_{i-1},\mathbf{y}^{i-1}\right),\ \text{if } t_k^{i-1}=T_i, \end{cases} \tag{9.53}$$

$$\begin{cases} \mathbb{E}\left[\Delta\left(p_j\right)^q\left(t_k^{i-1}\right)|\mathcal{F}_{k-1}^{i-1}\right]= \\ \left[q\left(p_j\right)^q\left(t_{k-1}^{i-1}\right)\left(\gamma_{j,j}^{i-1}(y_j(t_{k-1}^{i-1})-p_j(t_{k-1}^{i-1}))+\beta_j^{i-1}+\sum\limits_{l\neq j}^{n}\gamma_{j,l}^{i-1}p_l(t_{k-1}^{i-1})\right)\right. \\ \left.+\frac{q(q-1)}{2\Delta t_i}p_j^{q-2}(t_{k-1}^{i-1})\mathbb{E}\left[\left(\Delta p_j(t_k^{i-1})-\mathbb{E}[\Delta p_j(t_k^{i-1})|\mathcal{F}_{k-1}^{i-1}]\right)^2|\mathcal{F}_{k-1}^{i-1}\right]\right]\Delta t, \\ \text{for } t_k^{i-1}\in[T_{i-1},T_i), \\ \mathbb{E}\left[\left(p_j^i\right)^q|\mathcal{F}_{i-1}\right]=\left(\theta_j^i\right)^q\left(p_j\right)^q\left(T_i^-,T_{i-1},\mathbf{y}^{i-1},\mathbf{p}^{i-1}\right),\ \text{if } t_k^{i-1}=T_i,\ q\in I(1,n+1), \end{cases} \tag{9.54}$$

$$\begin{cases} \mathbb{E}\left[\left(\Delta\left(y_j\right)^q\left(t_k^{i-1}\right)-\mathbb{E}[\Delta\left(y_j\right)^q\left(t_k^{i-1}\right)|\mathcal{F}_{k-1}^{i-1}]\right)\times\right. \\ \left.\left(\Delta\left(y_l\right)^q\left(t_k^{i-1}\right)-\mathbb{E}[\Delta\left(y_l\right)^q\left(t_k^{i-1}\right)|\mathcal{F}_{k-1}^{i-1}]\right)|\mathcal{F}_{k-1}^{i-1}\right]= \\ q^2\left(y_jy_l\right)^{q-1}\left(t_{k-1}^{i-1}\right)\left(u_j^{i-1}-y_j(t_{k-1}^{i-1})\right)\left(u_l^{i-1}-y_l(t_{k-1}^{i-1})\right)\left[\delta_{j,j}^{i-1}\delta_{l,j}^{i-1}y_j(t_{k-1}^{i-1})\right. \\ \left.+\delta_{l,l}^{i-1}\delta_{j,l}^{i-1}y_l(t_{k-1}^{i-1})+\sum\limits_{r=1,j,l\neq r}^{n}\delta_{j,r}^{i-1}\delta_{l,r}^{i-1}y_r^2(t_{k-1}^{i-1})\right]\Delta t, \\ t_k^{i-1}\in[T_{i-1}-\tau_{i-1},T_i), \end{cases} \tag{9.55}$$

$$\begin{cases} \mathbb{E}\left[\left(\Delta\left(p_j\right)^q\left(t_k^{i-1}\right)-\mathbb{E}[\Delta\left(p_j\right)^q\left(t_k^{i-1}\right)|\mathcal{F}_{k-1}^{i-1}]\right)\times \right. \\ \left.\left(\Delta\left(p_l\right)^q\left(t_k^{i-1}\right)-\mathbb{E}[\Delta\left(p_l\right)^q\left(t_k^{i-1}\right)|\mathcal{F}_{k-1}^{i-1}]\right)|\mathcal{F}_{k-1}^{i-1}\right]= \\ q^2\left(p_jp_l\right)^q\left(t_{k-1}^{i-1}\right)\left[2\sigma_{j,j}^{i-1}\sigma_{l,j}^{i-1}y_j(t_{k-1}^{i-1})+\sum_{r=1,j,l\neq r}^{n}\sigma_{j,r}^{i-1}\sigma_{l,r}^{i-1}y_r^2(t_{k-1}^{i-1})\right], \\ j\neq l, q\in I(1,2n) \end{cases} \tag{9.56}$$

where $\mathcal{F}_{k-1}^{i-1}$ is the filtration up to time t_{k-1}^{i-1}. From (9.53)-(9.56), (9.51) reduces to

$$\begin{cases} \Delta\left(y_j\right)^q\left(t_k^{i-1}\right) &= \mathbb{E}\left[\Delta\left(y_j\right)^q\left(t_k^{i-1}\right)|\mathcal{F}_{k-1}^{i-1}\right] \\ & \quad +q\left(y_j\right)^{q-1}\left(t_{k-1}^{i-1}\right)\left(u_j^{i-1}-y_j(t_{k-1}^{i-1})\right)\left[\delta_{j,j}{}^{i-1}\Delta W_{j,j}(t_k^{i-1})\right. \\ & \quad \left.+\sum_{l\neq j}^{n}\delta_{j,l}{}^{i-1}y_l\Delta W_{j,l}(t_k^{i-1})\right],\ y_j\left(T_{i-1}\right)=y_j^{i-1},\ t_k^{i-1}\in[T_{i-1},T_i), \\ \left(y_j^i\right)^q &= \left(\pi_j^i\right)^q\left(y_j\right)^q\left(T_i^-,T_{i-1},\mathbf{y}^{i-1}\right),\ \text{if } t_k^{i-1}=T_i \\ \Delta\left(p_j\right)^q\left(t_k^{i-1}\right) &= \mathbb{E}\left[\Delta\left(p_j\right)^q\left(t_k^{i-1}\right)|\mathcal{F}_{k-1}^{i-1}\right] \\ & \quad +q\left(p_j\right)^q\left(t_{k-1}^{i-1}\right)\left[\sigma_{j,j}^{i-1}\Delta Z_{j,j}(t_k^{i-1})+\sum_{l\neq j}^{n}\sigma_{j,l}^{i-1}p_l(t_{k-1}^{i-1})\Delta Z_{j,l}(t_k^{i-1})\right], \\ & \quad p_j\left(T_{i-1}\right)=p_j^{i-1},\ t_k^{i-1}\in[T_{i-1},T_i),\ q\in I(1,n+1),\ j\in I(1,n), \\ \left(p_j^i\right)^q &= \left(\theta_j^i\right)^q\left(p_j\right)^q\left(T_i^-,T_{i-1},\mathbf{y}^{i-1},\mathbf{p}^{i-1}\right),\ \text{if } t_k^{i-1}=T_i \end{cases} \tag{9.57}$$

(9.57) provides the basis for the development of the concept of lagged adaptive expectation process [88] with respect to continuous time stochastic dynamic systems (9.45) and (9.50).

For $k \in I(0,N_{i-1})$, applying the lagged adaptive expectation process [88], from Definitions $9.5.1-9.5.3$, and using (9.53)-(9.57), we formulate a local observation/measurement process at t_k^{i-1} as a algebraic functions of m_k^{i-1}-local functions of restriction of the finite sample sequence $\{\mathbf{y}(t_l^{i-1})\}_{l=-r_{i-1}}^{N_{i-1}}$ and $\{\mathbf{p}(t_l^{i-1})\}_{l=-r_{i-1}}^{N_{i-1}}$ to subpartition P_k^{i-1} in Definition 9.5.2 :

$$\sum_{\iota=k-m_k^{i-1}}^{k-1} \mathbb{E}\left[\Delta\,(y_j)^q\,(t_\iota^{i-1})|\mathcal{F}_{\iota-1}^{i-1}\right] = \left\{ \begin{array}{l} \sum_{\iota=k-m_k^{i-1}}^{k-1} \left[q\,(y_j)^{q-1}\,(t_{\iota-1}^{i-1})\left(u_j^{i-1}-y_j(t_{\iota-1}^{i-1})\right)\times\right. \\ \left(\kappa_{j,j}^{i-1}y_j(t_{\iota-1}^{i-1})+\sum_{l\neq j}^{n}\kappa_{j,l}^{i-1}y_l(t_{\iota-1}^{i-1})\right) \\ \left.+\frac{q(q-1)}{2\Delta t}\,(y_j)^{q-2}\,(t_{\iota-1}^{i-1})s^{j,j}_{m_k^{i-1},k,\Delta y_j}\right]\Delta t, \end{array}\right.$$

$$\sum_{\iota=k-m_k^{i-1}}^{k-1} \mathbb{E}\left[\Delta\,(p_j)^q\,(t_\iota^{i-1})|\mathcal{F}_{\iota-1}^{i-1}\right] = \left\{ \begin{array}{l} \sum_{\iota=k-m_k^{i-1}}^{k-1} \left[q\,(p_j)^{q}\,(t_{\iota-1}^{i-1})\left(\gamma_{j,j}^{i-1}(y_j(t_{\iota-1}^{i-1})-p_j(t_{\iota-1}^{i-1}))\right.\right. \\ \left.+\beta_j^{i-1}+\sum_{l\neq j}^{n}\gamma_{j,l}p_l(t_{\iota-1}^{i-1})\right) \\ \left.+\frac{q(q-1)}{2\Delta t}p_j^{q-2}(t_{\iota-1}^{i-1})s^{j,j}_{m_k^{i-1},k,\Delta p_j}\right]\Delta t,\;\; q\in I(1,n+1), \end{array}\right. \tag{9.58}$$

and

$$\left\{ \begin{array}{rcl} \hat{s}^{j,l}_{m_k^{i-1},k}(\Delta\,(y)^q) & = & \frac{1}{m_k^{i-1}}\sum_{\iota=k-m_k^{i-1}}^{k-1} q^2\,(y_jy_l)^{q-1}\,(t_{\iota-1}^{i-1})\left(u_j^{i-1}-y_j(t_{\iota-1}^{i-1})\right)\left(u_l^{i-1}-y_l(t_{\iota-1}^{i-1})\right)\times \\ & & \left[\delta_{j,j}^{i-1}\delta_{l,j}^{i-1}y_j(t_{\iota-1}^{i-1})+\delta_{l,l}^{i-1}\delta_{j,l}^{i-1}y_l(t_{k-1}^{i-1})+\sum_{\substack{r=1\\ j\neq l\neq r}}^{n}\delta_{j,r}^{i-1}\delta_{l,r}^{i-1}y_r^2(t_{\iota-1}^{i-1})\right], \\ \hat{s}^{j,l}_{m_k^{i-1},k}(\Delta\,(p)^q) & = & \frac{1}{m_k^{i-1}}\sum_{\iota=k-m_k^{i-1}}^{k-1} q^2\,(p_jp_l)^{q}\,(t_{\iota-1}^{i-1})\left[\sigma_{j,j}^{i-1}\sigma_{l,j}^{i-1}p_j(t_{\iota-1}^{i-1})+\sigma_{l,l}^{i-1}\sigma_{j,l}^{i-1}p_l(t_{\iota-1}^{i-1})\right. \\ & & \left.+\sum_{\substack{r=1\\ j\neq l\neq r}}^{n}\sigma_{j,r}^{i-1}\sigma_{l,r}^{i-1}p_r^2(t_{\iota-1}^{i-1})\right],\; j\neq l,\; q\in I(1,2n). \end{array}\right. \tag{9.59}$$

For each $i\in I(1,K^*)$ and each $j\neq\mathfrak{r}\in I(1,n)$, we define

$$\begin{array}{rcl} F_{1q}\left(u_j^{i-1},\left\{\kappa_{j,\mathfrak{r}}^{i-1}\right\}_{\mathfrak{r}=1}^n\right) & \equiv & F_{1q}\left(\mathbb{E}\left[\Delta\left(y_j^q\right)(t_\iota^{i-1})|\mathcal{F}_{\iota-1}^{i-1}\right];u_j^{i-1},\left\{\kappa_{j,\mathfrak{r}}^{i-1}\right\}_{\mathfrak{r}=1}^n\right), \\ F_{2q}\left(\left\{\delta_{j,\mathfrak{r}}^{i-1},\delta_{l,\mathfrak{r}}^{i-1}\right\}_{\mathfrak{r}=1}^n\right) & \equiv & F_{2q}\left(\mathbb{E}\left[\Delta\left(y_j^q\right)(t_\iota^{i-1})|\mathcal{F}_{\iota-1}^{i-1}\right],\mathbb{E}\left[\Delta\left(y_l^q\right)(t_\iota^{i-1})|\mathcal{F}_{\iota-1}^{i-1}\right];\left\{\delta_{j,\mathfrak{r}}^{i-1},\delta_{l,\mathfrak{r}}^{i-1}\right\}_{\mathfrak{r}=1}^n\right), \\ G_{1q}\left(\beta_j^{i-1},\left\{\gamma_{j,\mathfrak{r}}^{i-1}\right\}_{\mathfrak{r}=1}^n\right) & \equiv & G_{1q}\left(\mathbb{E}\left[\Delta\left(p_j^q\right)(t_\iota^{i-1})|\mathcal{F}_{\iota-1}^{i-1}\right];\beta_j^{i-1},\left\{\gamma_{j,\mathfrak{r}}^{i-1}\right\}_{\mathfrak{r}=1}^n\right) \\ G_{2q}\left(\left\{\sigma_{j,\mathfrak{r}}^{i-1},\sigma_{l,\mathfrak{r}}^{i-1}\right\}_{\mathfrak{r}=1}^n\right) & \equiv & G_{2q}\left(\mathbb{E}\left[\Delta\left(p_j^q\right)(t_\iota^{i-1})|\mathcal{F}_{\iota-1}^{i-1}\right],\mathbb{E}\left[\Delta\left(p_l^q\right)(t_\iota^{i-1})|\mathcal{F}_{\iota-1}^{i-1}\right];\left\{\sigma_{j,\mathfrak{r}},\sigma_{l,\mathfrak{r}}^{i-1}\right\}_{\mathfrak{r}=1}^n\right), \end{array}$$

by

$$
\begin{aligned}
F_{1q}\left(u_j^{i-1},\left\{\kappa_{j,\mathfrak{r}}^{i-1}\right\}_{\mathfrak{r}=1}^{n}\right) &= \tfrac{1}{m_k^{i-1}}\sum_{\iota=k-m_k^{i-1}}^{k-1}\Big\{\Big[q\left(y_j\right)^{q-1}(t_{\iota-1}^{i-1})\left(u_j^{i-1}-y_j(t_{\iota-1}^{i-1})\right)\times\\
&\quad\left(\sum_{\mathfrak{r}=1}^{n}\kappa_{j,\mathfrak{r}}y_{\mathfrak{r}}(t_{\iota-1}^{i-1})\right)\\
&\quad+\tfrac{q(q-1)}{2\Delta t}\left(y_j\right)^{q-2}(t_{\iota-1}^{i-1})\hat{s}^{j,j}_{m_k^{i-1},k}(\Delta y_j)\Big]\,\Delta t\Big\}\\
&\quad-\tfrac{1}{m_k^{i-1}}\sum_{\iota=k-m_k^{i-1}}^{k-1}\mathbb{E}\left[\Delta\left(y_j\right)^q(t_\iota^{i-1})|\mathcal{F}_{\iota-1}^{i-1}\right],\ q\in I(1,n+1),\\
F_{2q}\left(\left\{\delta_{j,\mathfrak{r}}^{i-1},\delta_{l,\mathfrak{r}}^{i-1}\right\}_{\mathfrak{r}=1}^{n}\right) &= \tfrac{1}{m_k^{i-1}}\sum_{\iota=k-m_k^{i-1}}^{k-1}q^2\left(y_jy_l\right)^{q-1}(t_{\iota-1}^{i-1})\left(u_j^{i-1}-y_j(t_{\iota-1}^{i-1})\right)\times\\
&\quad\left(u_l^{i-1}-y_l(t_{\iota-1}^{i-1})\right)\Big[\delta_{j,j}^{i-1}\delta_{l,j}^{i-1}y_j(t_{\iota-1}^{i-1})+\delta_{l,l}^{i-1}\delta_{j,l}^{i-1}y_l(t_{\iota-1}^{i-1})\\
&\quad+\sum_{\substack{\mathfrak{r}=1\\ j\neq l\neq\mathfrak{r}}}^{n}\delta_{j,\mathfrak{r}}^{i-1}\delta_{l,\mathfrak{r}}^{i-1}y_{\mathfrak{r}}^2(t_{\iota-1}^{i-1})\Big]-\hat{s}^{j,l}_{m_k^{i-1},k,}(\Delta\left(y\right)^q),\\
&\quad j\neq l\in I(1,n),\ q\in I(1,2n)\\
G_{1q}\left(\beta_j^{i-1},\left\{\gamma_{j,\mathfrak{r}}^{i-1}\right\}_{\mathfrak{r}=1}^{n}\right) &= \tfrac{1}{m_k^{i-1}}\sum_{\iota=k-m_k^{i-1}}^{k-1}\Big\{\Big[q\left(p_j\right)^{q}(t_{\iota-1}^{i-1})\Big(\gamma_{j,j}^{i-1}(y_j(t_{\iota-1}^{i-1})-p_j(t_{\iota-1}^{i-1}))\\
&\quad+\beta_j^{i-1}+\sum_{\mathfrak{r}\neq j}^{n}\gamma_{j,\mathfrak{r}}^{i-1}p_{\mathfrak{r}}(t_{\iota-1}^{i-1})\Big)\\
&\quad+\tfrac{q(q-1)}{2\Delta t}p_j^{q-2}(t_{\iota-1}^{i-1})\hat{s}^{j,j}_{m_k^{i-1},k}(\Delta p_j)\Big]\,\Delta t\Big\}\\
&\quad-\tfrac{1}{m_k^{i-1}}\sum_{\iota=k-m_k^{i-1}}^{k-1}\mathbb{E}\left[\Delta\left(p_j\right)^q(t_\iota^{i-1})|\mathcal{F}_{\iota-1}^{i-1}\right],\ \ q\in I(1,n+1),\\
G_{2q}\left(\left\{\sigma_{j,\mathfrak{r}}^{i-1},\sigma_{l,\mathfrak{r}}^{i-1}\right\}_{\mathfrak{r}=1}^{n}\right) &= \tfrac{1}{m_k^{i-1}}\sum_{\iota=k-m_k^{i-1}}^{k-1}q^2\left(p_jp_l\right)^{q}(t_{\iota-1}^{i-1})\Big[\sigma_{j,j}^{i-1}\sigma_{l,j}^{i-1}p_j(t_{\iota-1}^{i-1})\\
&\quad+\sigma_{l,l}^{i-1}\sigma_{j,l}^{i-1}p_l(t_{\iota-1}^{i-1})+\sum_{\substack{\mathfrak{r}=1\\ j\neq l\neq\mathfrak{r}}}^{n}\sigma_{j,\mathfrak{r}}^{i-1}\sigma_{l,\mathfrak{r}}^{i-1}p_{\mathfrak{r}}^2(t_{\iota-1}^{i-1})\Big]\\
&\quad-\hat{s}^{j,l}_{m_k^{i-1},k,\Delta(p)^q},\ j\neq l\in I(1,n),\ \ q\in I(1,2n).
\end{aligned}
\tag{9.60}
$$

For every $j\in I(1,n)$, we have

$$
\begin{cases}
F_{1q}\left(u_j^{i-1},\left\{\kappa_{j,\mathfrak{r}}^{i-1}\right\}_{\mathfrak{r}=1}^{n}\right) = 0,\ q\in I(1,n+1),\\
F_{2q}\left(\left\{\delta_{j,\mathfrak{r}}^{i-1},\delta_{l,\mathfrak{r}}^{i-1}\right\}_{\mathfrak{r}=1}^{n}\right) = 0,\ q\in I(1,2n),\\
G_{1q}\left(\beta_j^{i-1},\left\{\gamma_{j,\mathfrak{r}}^{i-1}\right\}_{\mathfrak{r}=1}^{n}\right) = 0,\ q\in I(1,n+1),\\
F_{2q}\left(\left\{\delta_{j,\mathfrak{r}}^{i-1},\delta_{l,\mathfrak{r}}^{i-1}\right\}_{\mathfrak{r}=1}^{n}\right) = 0,\ q\in I(1,2n).
\end{cases}
\tag{9.61}
$$

Let us define $F_1=\{F_{1q}\}_{q\in I(1,n+1)}$, $F_2=\{F_{2q}\}_{q\in I(1,2n)}$, $G_1=\{G_{1q}\}_{q\in I(1,n+1)}$, and $G_2=\{G_{2q}\}_{q\in I(1,2n)}$.

Thus, provided that the determinant of each of the Jacobian matrices $JF_1\left(u_j^{i-1},\left\{\kappa_{j,\mathfrak{r}}^{i-1}\right\}_{\mathfrak{r}=1}^{n}\right)$, $JF_2\left(u_j^{i-1},\left\{\delta_{j,l}^{i-1}\right\}_{l\in I(1,n)}\right)$, $JG_1\left(\beta_j^{i-1},\left\{\gamma_{j,\mathfrak{r}}^{i-1}\right\}_{\mathfrak{r}=1}^{n}\right)$ and $JG_2\left(\left\{\sigma_{j,\mathfrak{r}}^{i-1},\sigma_{l,\mathfrak{r}}^{i-1}\right\}_{\mathfrak{r}=1}^{n}\right)$ are not zero, by the application of Theorem 9.1 (Implicit Function Theorem), we conclude that for every non-constant m_k^{i-1}-local sequence $\{y_j(t_l^{i-1})\}_{l=k-m_k^{i-1}}^{k-1}$ and $\{p_j(t_l^{i-1})\}_{l=k-m_k^{i-1}}^{k-1}$, there exist a unique solution $(\hat{u}_j^{i-1}(m_k^{i-1},t_k^{i-1})$, $\hat{\beta}_j^{i-1}(m_k^{i-1},t_k^{i-1})$, $\{\hat{\kappa}_{j,\mathfrak{r}}^{i-1}\}_{\mathfrak{r}=1}^{n}(m_k^{i-1},t_k^{i-1})$, $\{\hat{\gamma}_{j,\mathfrak{r}}^{i-1}\}_{\mathfrak{r}=1}^{n}(m_k^{i-1},t_k^{i-1})$, $\{\hat{\delta}_{j,\mathfrak{r}}^{i-1}\}_{\mathfrak{r}=1}^{n}(m_k^{i-1},t_k^{i-1})$, $\{\hat{\sigma}_{j,\mathfrak{r}}^{i-1}\}_{\mathfrak{r}=1}^{n}(m_k^{i-1},t_k^{i-1}))$ of system of algebraic equations (9.63) as a point estimates of u_j^{i-1}, $\{\kappa_{j,\mathfrak{r}}^{i-1}\}_{\mathfrak{r}=1}^{n}$, $\{\gamma_{j,\mathfrak{r}}^{i-1}\}_{\mathfrak{r}=1}^{n}$, $\{\delta_{j,\mathfrak{r}}^{i-1}\}_{\mathfrak{r}=1}^{n}$, $\{\sigma_{j,\mathfrak{r}}^{i-1}\}_{\mathfrak{r}=1}^{n}$, $j\in I(1,n)$, respectively. In the next section, illustrating this approach using energy commodities natural gas, crude oil and coal [26, 27, 28], we show conditions in which the determinant of the Jacobian matrix is not zero.

9.6.2 Illustration: Application to Energy Commodity

In this subsection, we present an illustration regarding the natural gas, crude oil and coal [26, 27, 28]. Here, $j\in I(1,3)$ and $i\in I(1,K^*)$. Moreover, (9.45) reduces to

$$\begin{cases} dy_j &= \left(u_j^{i-1}-y_j\right)\left[\kappa_{j,j}^{i-1}y_j+\sum\limits_{l\neq j}^{3}\kappa_{j,l}^{i-1}y_l\right]dt+\delta_{j,j}^{i-1}\left(u_j^{i-1}-y_j\right)dW_{j,j}(t) \\ & \quad +\left(u_j^{i-1}-y_j\right)\sum\limits_{l\neq j}^{3}\delta_{j,l}^{i-1}y_l dW_{j,l}(t),\ y_j\left(T_{i-1}\right)=y_j^{i-1},\ t\in[T_{i-1},T_i), \\ y_j^i &= \pi_j^i y_j\left(T_i^-,T_{i-1},\mathbf{y}^{i-1}\right), \\ dp_j(t) &= p_j\left[\gamma_{j,j}^{i-1}\left(y_j-p_j\right)+\beta_j^{i-1}+\sum\limits_{l\neq j}^{3}\gamma_{j,l}^{i-1}p_l(t)\right]dt+\sigma_{j,j}^{i-1}p_j dZ_{j,j}(t) \\ & \quad +p_j\sum\limits_{l\neq j}^{3}\sigma_{j,l}^{i-1}p_l dZ_{j,l}(t),\ p_j(T_{i-1})=p_j^{i-1},\ t\in[T_{i-1},T_i), \\ p_j^i &= \theta_j^i p_j\left(T_i^-,T_{i-1},\mathbf{y}^{i-1},\mathbf{p}^{i-1}\right). \end{cases} \tag{9.62}$$

For each $j\in I(1,3)$, following the argument used in Illustration 9.6.1, we have

$$\begin{aligned} F_{1q}\left(u_j^{i-1},\left\{\kappa_{j,\mathfrak{r}}^{i-1}\right\}_{\mathfrak{r}=1}^{3}\right) &= \frac{1}{m_k^{i-1}}\sum_{\iota=k-m_k^{i-1}}^{k-1}\left\{\left[q\left(y_j\right)^{q-1}\left(t_{\iota-1}^{i-1}\right)\left(u_j^{i-1}-y_j(t_{\iota-1}^{i-1})\right)\times\right.\right. \\ & \left(\sum_{\mathfrak{r}=1}^{3}\kappa_{j,\mathfrak{r}}y_{\mathfrak{r}}(t_{\iota-1}^{i-1})\right) \\ & \left.\left.+\frac{q(q-1)}{2\Delta t}\left(y_j\right)^{q-2}\left(t_{\iota-1}^{i-1}\right)\hat{s}_{m_k^{i-1},k}^{j,j}(\Delta y_j)\right]\Delta t\right\} \end{aligned}$$

$$
\begin{aligned}
& -\frac{1}{m_k^{i-1}} \sum_{\iota=k-m_k^{i-1}}^{k-1} \mathbb{E}\left[\Delta\left(y_j\right)^q\left(t_\iota^{i-1}\right) | \mathcal{F}_{\iota-1}^{i-1}\right], \ q \in I(1,4), \\
F_{2q}\left(\left\{\delta_{j,\mathfrak{r}}^{i-1}, \delta_{l,\mathfrak{r}}^{i-1}\right\}_{\mathfrak{r}=1}^3\right) \ = \ & \frac{1}{m_k^{i-1}} \sum_{\iota=k-m_k^{i-1}}^{k-1} q^2\left(y_j y_l\right)^{q-1}\left(t_{\iota-1}^{i-1}\right)\left(u_j^{i-1}-y_j(t_{\iota-1}^{i-1})\right) \times \\
& \left(u_l^{i-1}-y_l(t_{\iota-1}^{i-1})\right)\left[\delta_{j,j}^{i-1} \delta_{l,j}^{i-1} y_j(t_{\iota-1}^{i-1})+\delta_{l,l}^{i-1} \delta_{j,l}^{i-1} y_l(t_{\iota-1}^{i-1})\right. \\
& \left.+\sum_{\substack{\mathfrak{r}=1 \\ j \neq l \neq \mathfrak{r}}}^3 \delta_{j,\mathfrak{r}}^{i-1} \delta_{l,\mathfrak{r}}^{i-1} y_{\mathfrak{r}}^2(t_{\iota-1}^{i-1})\right]-\hat{s}_{m_k^{i-1},k,}^{j,l}(\Delta\left(y\right)^q), \\
& j \neq l \in I(1,3), \ q \in I(1,6) \\
G_{1q}\left(\beta_j^{i-1},\left\{\gamma_{j,\mathfrak{r}}^{i-1}\right\}_{\mathfrak{r}=1}^3\right) \ = \ & \frac{1}{m_k^{i-1}} \sum_{\iota=k-m_k^{i-1}}^{k-1}\left\{\left[q\left(p_j\right)^q\left(t_{\iota-1}^{i-1}\right)\left(\gamma_{j,j}^{i-1}(y_j(t_{\iota-1}^{i-1})-p_j(t_{\iota-1}^{i-1}))\right.\right.\right. \\
& \left.+\beta_j^{i-1}+\sum_{\mathfrak{r} \neq j}^3 \gamma_{j,\mathfrak{r}}^{i-1} p_{\mathfrak{r}}(t_{\iota-1}^{i-1})\right) \\
& \left.\left.+\frac{q(q-1)}{2 \Delta t} p_j^{q-2}(t_{\iota-1}^{i-1}) \hat{s}_{m_k^{i-1},k}^{j,j}(\Delta p_j)\right] \Delta t\right\} \\
& -\frac{1}{m_k^{i-1}} \sum_{\iota=k-m_k^{i-1}}^{k-1} \mathbb{E}\left[\Delta\left(p_j\right)^q\left(t_\iota^{i-1}\right) | \mathcal{F}_{\iota-1}^{i-1}\right], \ \ q \in I(1,4), \\
G_{2q}\left(\left\{\sigma_{j,\mathfrak{r}}^{i-1}, \sigma_{l,\mathfrak{r}}^{i-1}\right\}_{\mathfrak{r}=1}^3\right) \ = \ & \frac{1}{m_k^{i-1}} \sum_{\iota=k-m_k^{i-1}}^{k-1} q^2\left(p_j p_l\right)^q\left(t_{\iota-1}^{i-1}\right)\left[\sigma_{j,j}^{i-1} \sigma_{l,j}^{i-1} p_j(t_{\iota-1}^{i-1})\right. \\
& \left.+\sigma_{l,l}^{i-1} \sigma_{j,l}^{i-1} p_l(t_{\iota-1}^{i-1})+\sum_{\substack{\mathfrak{r}=1 \\ j \neq l \neq \mathfrak{r}}}^3 \sigma_{j,\mathfrak{r}}^{i-1} \sigma_{l,\mathfrak{r}}^{i-1} p_{\mathfrak{r}}^2(t_{\iota-1}^{i-1})\right] \\
& -\hat{s}_{m_k^{i-1},k,\Delta(p)^q}^{j,l}, \ j \neq l \in I(1,3), \ \ q \in I(1,6).
\end{aligned}
$$

and for $j \neq l \in I(1,3)$, we have

$$
\left\{\begin{array}{lll}
F_{1q}\left(u_j^{i-1},\left\{\kappa_{j,\mathfrak{r}}^{i-1}\right\}_{\mathfrak{r}=1}^3\right) & = & 0, \ \ q \in I(1,4), \\
F_{2q}\left(\left\{\delta_{j,\mathfrak{r}}^{i-1}, \delta_{l,\mathfrak{r}}^{i-1}\right\}_{\mathfrak{r}=1}^3\right) & = & 0, \ \ q \in I(1,6), \\
G_{1q}\left(\beta_j^{i-1},\left\{\gamma_{j,\mathfrak{r}}^{i-1}\right\}_{\mathfrak{r}=1}^3\right) & = & 0, \ \ q \in I(1,4), \\
F_{2q}\left(\left\{\delta_{j,\mathfrak{r}}^{i-1}, \delta_{l,\mathfrak{r}}^{i-1}\right\}_{\mathfrak{r}=1}^3\right) & = & 0, \ \ q \in I(1,6).
\end{array}\right. \tag{9.63}
$$

We also $F_1=\{F_{1q}\}_{q \in I(1,4)}$, $F_2=\{F_{2q}\}_{q \in I(1,3)}$, $G_1=\{G_{1q}\}_{q \in I(1,4)}$, and $G_2=\{G_{2q}\}_{q \in I(1,3)}$. Thus, for each $j \in I(1,3)$, the determinant of the Jacobian matrix $JF_1\left(u_j^{i-1},\left\{\kappa_{j,\mathfrak{r}}^{i-1}\right\}_{\mathfrak{r}=1}^n\right)$ is given by $\frac{1}{\left(m_k^{i-1}\right)^4} det \left[\sum\limits_{\iota=k-m_k^{i-1}}^{k-1} \mathcal{J}_\iota\right]$, where $\mathcal{J}_\iota$ is define by $\mathcal{J}_\iota =$

$$\begin{pmatrix} \sum\limits_{\mathfrak{r}=1}^{n} \kappa_{j,\mathfrak{r}} y_\mathfrak{r}(t_{\iota-1}^{i-1}) & (u_j - y_j(t_{\iota-1}^{i-1}))y_1(t_{\iota-1}^{i-1}) & (u_j - y_j(t_{\iota-1}^{i-1}))y_2(t_{\iota-1}^{i-1}) & (u_j - y_j(t_{\iota-1}^{i-1}))y_3(t_{\iota-1}^{i-1}) \\ 2\sum\limits_{\mathfrak{r}=1}^{n} \kappa_{j,\mathfrak{r}} y_\mathfrak{r}(t_{\iota-1}^{i-1}) y_j(t_{\iota-1}^{i-1}) & 2(u_j - y_j(t_{\iota-1}^{i-1}))y_j(t_{\iota-1}^{i-1})y_1(t_{\iota-1}^{i-1}) & 2(u_j - y_j(t_{\iota-1}^{i-1}))y_j(t_{\iota-1}^{i-1})y_2(t_{\iota-1}^{i-1}) & 2(u_j - y_j(t_{\iota-1}^{i-1}))y_j(t_{\iota-1}^{i-1})y_3(t_{\iota-1}^{i-1}) \\ 3\sum\limits_{\mathfrak{r}=1}^{n} \kappa_{j,\mathfrak{r}} y_\mathfrak{r}(t_{\iota-1}^{i-1}) y_j^2(t_{\iota-1}^{i-1}) & 3(u_j - y_j(t_{\iota-1}^{i-1}))y_j^2(t_{\iota-1}^{i-1})y_1(t_{\iota-1}^{i-1}) & 3(u_j - y_j(t_{\iota-1}^{i-1}))y_j^2(t_{\iota-1}^{i-1})y_2(t_{\iota-1}^{i-1}) & 3(u_j - y_j(t_{\iota-1}^{i-1}))y_j^2(t_{\iota-1}^{i-1})y_3(t_{\iota-1}^{i-1}) \\ 4\sum\limits_{\mathfrak{r}=1}^{n} \kappa_{j,\mathfrak{r}} y_\mathfrak{r}(t_{\iota-1}^{i-1}) y_j^3(t_{\iota-1}^{i-1}) & 4(u_j - y_j(t_{\iota-1}^{i-1}))y_j^3(t_{\iota-1}^{i-1})y_1(t_{\iota-1}^{i-1}) & 4(u_j - y_j(t_{\iota-1}^{i-1}))y_j^3(t_{\iota-1}^{i-1})y_2(t_{\iota-1}^{i-1}) & 4(u_j - y_j(t_{\iota-1}^{i-1}))y_j^3(t_{\iota-1}^{i-1})y_3(t_{\iota-1}^{i-1}) \end{pmatrix}$$

and det $\left[\sum\limits_{\iota=k-m_k^{i-1}}^{k-1} \mathcal{J}_\iota\right]$

$$\begin{aligned}
= \quad & \sum_{\iota=k-m_k^{i-1}}^{k-1} \left[\sum_{\mathfrak{r}=1}^{n} \kappa_{j,\mathfrak{r}} y_\mathfrak{r}(t_{\iota-1}^{i-1})\right] y_j^3(t_{\iota-1}^{i-1}) \sum_{\iota=k-m_k^{i-1}}^{k-1} (u_j - y_j(t_{\iota-1}^{i-1}))y_j^2(t_{\iota-1}^{i-1})y_1(t_{\iota-1}^{i-1}) \times \\
& \left(\sum_{\iota=k-m_k^{i-1}}^{k-1} (u_j - y_j(t_{\iota-1}^{i-1}))y_j^3(t_{\iota-1}^{i-1})y_3(t_{\iota-1}^{i-1}) \sum_{\iota=k-m_k^{i-1}}^{k-1} (u_j - y_j(t_{\iota-1}^{i-1}))y_j(t_{\iota-1}^{i-1})y_2(t_{\iota-1}^{i-1}) \right. \\
& \left. - \sum_{\iota=k-m_k^{i-1}}^{k-1} (u_j - y_j(t_{\iota-1}^{i-1}))y_2(t_{\iota-1}^{i-1}) \sum_{\iota=k-m_k^{i-1}}^{k-1} (u_j - y_j(t_{\iota-1}^{i-1}))y_j(t_{\iota-1}^{i-1})y_3(t_{\iota-1}^{i-1}) \right) \\
+ \quad & \sum_{\iota=k-m_k^{i-1}}^{k-1} \left[\sum_{\mathfrak{r}=1}^{n} \kappa_{j,\mathfrak{r}} y_\mathfrak{r}(t_{\iota-1}^{i-1})\right] y_j^3(t_{\iota-1}^{i-1}) \sum_{\iota=k-m_k^{i-1}}^{k-1} (u_j - y_j(t_{\iota-1}^{i-1}))y_j^2(t_{\iota-1}^{i-1})y_1(t_{\iota-1}^{i-1}) \times \\
& \left(\sum_{\iota=k-m_k^{i-1}}^{k-1} (u_j - y_j(t_{\iota-1}^{i-1}))y_j(t_{\iota-1}^{i-1})y_3(t_{\iota-1}^{i-1}) \sum_{\iota=k-m_k^{i-1}}^{k-1} (u_j - y_j(t_{\iota-1}^{i-1}))y_1(t_{\iota-1}^{i-1}) \right. \\
& \left. - \sum_{\iota=k-m_k^{i-1}}^{k-1} (u_j - y_j(t_{\iota-1}^{i-1}))y_j(t_{\iota-1}^{i-1})y_1(t_{\iota-1}^{i-1}) \sum_{\iota=k-m_k^{i-1}}^{k-1} (u_j - y_j(t_{\iota-1}^{i-1}))y_3(t_{\iota-1}^{i-1}) \right) \\
+ \quad & \sum_{\iota=k-m_k^{i-1}}^{k-1} \left[\sum_{\mathfrak{r}=1}^{n} \kappa_{j,\mathfrak{r}} y_\mathfrak{r}(t_{\iota-1}^{i-1})\right] y_j^3(t_{\iota-1}^{i-1}) \sum_{\iota=k-m_k^{i-1}}^{k-1} (u_j - y_j(t_{\iota-1}^{i-1}))y_j^2(t_{\iota-1}^{i-1})y_1(t_{\iota-1}^{i-1}) \times \\
& \left(\sum_{\iota=k-m_k^{i-1}}^{k-1} (u_j - y_j(t_{\iota-1}^{i-1}))y_2(t_{\iota-1}^{i-1}) \sum_{\iota=k-m_k^{i-1}}^{k-1} (u_j - y_j(t_{\iota-1}^{i-1}))y_j(t_{\iota-1}^{i-1})y_1(t_{\iota-1}^{i-1}) \right. \\
& \left. - \sum_{\iota=k-m_k^{i-1}}^{k-1} (u_j - y_j(t_{\iota-1}^{i-1}))y_1(t_{\iota-1}^{i-1}) \sum_{\iota=k-m_k^{i-1}}^{k-1} (u_j - y_j(t_{\iota-1}^{i-1}))y_j(t_{\iota-1}^{i-1})y_2(t_{\iota-1}^{i-1}) \right)
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{\iota=k-m_k^{i-1}}^{k-1} \left[\sum_{\mathfrak{r}=1}^{n} \kappa_{j,\mathfrak{r}} y_{\mathfrak{r}}(t_{\iota-1}^{i-1})\right] y_j^2(t_{\iota-1}^{i-1}) \sum_{\iota=k-m_k^{i-1}}^{k-1} (u_j - y_j(t_{\iota-1}^{i-1})) y_j^3(t_{\iota-1}^{i-1}) y_1(t_{\iota-1}^{i-1}) \times \\
&\quad \left(\sum_{\iota=k-m_k^{i-1}}^{k-1} (u_j - y_j(t_{\iota-1}^{i-1})) y_2(t_{\iota-1}^{i-1}) \sum_{\iota=k-m_k^{i-1}}^{k-1} (u_j - y_j(t_{\iota-1}^{i-1})) y_j(t_{\iota-1}^{i-1}) y_3(t_{\iota-1}^{i-1}) \right. \\
&\quad \left. - \sum_{\iota=k-m_k^{i-1}}^{k-1} (u_j - y_j(t_{\iota-1}^{i-1})) y_3(t_{\iota-1}^{i-1}) \sum_{\iota=k-m_k^{i-1}}^{k-1} (u_j - y_j(t_{\iota-1}^{i-1})) y_j(t_{\iota-1}^{i-1}) y_2(t_{\iota-1}^{i-1}) \right) \\
&+ \sum_{\iota=k-m_k^{i-1}}^{k-1} (u_j - y_j(t_{\iota-1}^{i-1})) y_j^2(t_{\iota-1}^{i-1}) y_2(t_{\iota-1}^{i-1}) \sum_{\iota=k-m_k^{i-1}}^{k-1} (u_j - y_j(t_{\iota-1}^{i-1})) y_j(t_{\iota-1}^{i-1}) y_1(t_{\iota-1}^{i-1}) \times \\
&\quad \left(\sum_{\iota=k-m_k^{i-1}}^{k-1} (u_j - y_j(t_{\iota-1}^{i-1})) y_3(t_{\iota-1}^{i-1}) \sum_{\iota=k-m_k^{i-1}}^{k-1} \left[\sum_{\mathfrak{r}=1}^{n} \kappa_{j,\mathfrak{r}} y_{\mathfrak{r}}(t_{\iota-1}^{i-1})\right] y_j(t_{\iota-1}^{i-1}) \right. \\
&\quad \left. - \sum_{\iota=k-m_k^{i-1}}^{k-1} \left[\sum_{\mathfrak{r}=1}^{n} \kappa_{j,\mathfrak{r}} y_{\mathfrak{r}}(t_{\iota-1}^{i-1})\right] \sum_{\iota=k-m_k^{i-1}}^{k-1} (u_j - y_j(t_{\iota-1}^{i-1})) y_j(t_{\iota-1}^{i-1}) y_3(t_{\iota-1}^{i-1}) \right) \\
&+ \sum_{\iota=k-m_k^{i-1}}^{k-1} (u_j - y_j(t_{\iota-1}^{i-1})) y_j^2(t_{\iota-1}^{i-1}) y_3(t_{\iota-1}^{i-1}) \sum_{\iota=k-m_k^{i-1}}^{k-1} (u_j - y_j(t_{\iota-1}^{i-1})) y_j^3(t_{\iota-1}^{i-1}) y_1(t_{\iota-1}^{i-1}) \times \\
&\quad \left(\sum_{\iota=k-m_k^{i-1}}^{k-1} \left[\sum_{\mathfrak{r}=1}^{n} \kappa_{j,\mathfrak{r}} y_{\mathfrak{r}}(t_{\iota-1}^{i-1})\right] \sum_{\iota=k-m_k^{i-1}}^{k-1} (u_j - y_j(t_{\iota-1}^{i-1})) y_j(t_{\iota-1}^{i-1}) y_2(t_{\iota-1}^{i-1}) \right. \\
&\quad \left. - \sum_{\iota=k-m_k^{i-1}}^{k-1} (u_j - y_j(t_{\iota-1}^{i-1})) y_2(t_{\iota-1}^{i-1}) \sum_{\iota=k-m_k^{i-1}}^{k-1} \left[\sum_{\mathfrak{r}=1}^{n} \kappa_{j,\mathfrak{r}} y_{\mathfrak{r}}(t_{\iota-1}^{i-1})\right] y_j(t_{\iota-1}^{i-1}) \right) \\
&+ \sum_{\iota=k-m_k^{i-1}}^{k-1} \left[\sum_{\mathfrak{r}=1}^{n} \kappa_{j,\mathfrak{r}} y_{\mathfrak{r}}(t_{\iota-1}^{i-1})\right] y_j^2(t_{\iota-1}^{i-1}) \sum_{\iota=k-m_k^{i-1}}^{k-1} (u_j - y_j(t_{\iota-1}^{i-1})) y_j^3(t_{\iota-1}^{i-1}) y_2(t_{\iota-1}^{i-1}) \times \\
&\quad \left(\sum_{\iota=k-m_k^{i-1}}^{k-1} (u_j - y_j(t_{\iota-1}^{i-1})) y_3(t_{\iota-1}^{i-1}) \sum_{\iota=k-m_k^{i-1}}^{k-1} (u_j - y_j(t_{\iota-1}^{i-1})) y_j(t_{\iota-1}^{i-1}) y_1(t_{\iota-1}^{i-1}) \right. \\
&\quad \left. - \sum_{\iota=k-m_k^{i-1}}^{k-1} (u_j - y_j(t_{\iota-1}^{i-1})) y_1(t_{\iota-1}^{i-1}) \sum_{\iota=k-m_k^{i-1}}^{k-1} (u_j - y_j(t_{\iota-1}^{i-1})) y_j(t_{\iota-1}^{i-1}) y_3(t_{\iota-1}^{i-1}) \right)
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{\iota=k-m_k^{i-1}}^{k-1} (u_j - y_j(t_{\iota-1}^{i-1}))y_j^2(t_{\iota-1}^{i-1})y_1(t_{\iota-1}^{i-1}) \sum_{\iota=k-m_k^{i-1}}^{k-1} (u_j - y_j(t_{\iota-1}^{i-1}))y_j^3(t_{\iota-1}^{i-1})y_2(t_{\iota-1}^{i-1}) \times \\
&\quad \left(\sum_{\iota=k-m_k^{i-1}}^{k-1} \left[\sum_{\mathfrak{r}=1}^{n} \kappa_{j,\mathfrak{r}} y_{\mathfrak{r}}(t_{\iota-1}^{i-1})\right] \sum_{\iota=k-m_k^{i-1}}^{k-1} (u_j - y_j(t_{\iota-1}^{i-1}))y_j(t_{\iota-1}^{i-1})y_3(t_{\iota-1}^{i-1}) \right. \\
&\quad \left. - \sum_{\iota=k-m_k^{i-1}}^{k-1} (u_j - y_j(t_{\iota-1}^{i-1}))y_3(t_{\iota-1}^{i-1}) \sum_{\iota=k-m_k^{i-1}}^{k-1} \left[\sum_{\mathfrak{r}=1}^{n} \kappa_{j,\mathfrak{r}} y_{\mathfrak{r}}(t_{\iota-1}^{i-1})\right] y_j(t_{\iota-1}^{i-1}) \right) \\
&+ \sum_{\iota=k-m_k^{i-1}}^{k-1} (u_j - y_j(t_{\iota-1}^{i-1}))y_j^2(t_{\iota-1}^{i-1})y_3(t_{\iota-1}^{i-1}) \sum_{\iota=k-m_k^{i-1}}^{k-1} (u_j - y_j(t_{\iota-1}^{i-1}))y_j^3(t_{\iota-1}^{i-1})y_2(t_{\iota-1}^{i-1}) \times \\
&\quad \left(\sum_{\iota=k-m_k^{i-1}}^{k-1} (u_j - y_j(t_{\iota-1}^{i-1}))y_1(t_{\iota-1}^{i-1}) \sum_{\iota=k-m_k^{i-1}}^{k-1} \left[\sum_{\mathfrak{r}=1}^{n} \kappa_{j,\mathfrak{r}} y_{\mathfrak{r}}(t_{\iota-1}^{i-1})\right] y_j(t_{\iota-1}^{i-1}) \right. \\
&\quad \left. - \sum_{\iota=k-m_k^{i-1}}^{k-1} \left[\sum_{\mathfrak{r}=1}^{n} \kappa_{j,\mathfrak{r}} y_{\mathfrak{r}}(t_{\iota-1}^{i-1})\right] \sum_{\iota=k-m_k^{i-1}}^{k-1} (u_j - y_j(t_{\iota-1}^{i-1}))y_j(t_{\iota-1}^{i-1})y_1(t_{\iota-1}^{i-1}) \right) \\
&+ \sum_{\iota=k-m_k^{i-1}}^{k-1} \left[\sum_{\mathfrak{r}=1}^{n} \kappa_{j,\mathfrak{r}} y_{\mathfrak{r}}(t_{\iota-1}^{i-1})\right] y_j^2(t_{\iota-1}^{i-1}) \sum_{\iota=k-m_k^{i-1}}^{k-1} (u_j - y_j(t_{\iota-1}^{i-1}))y_j^3(t_{\iota-1}^{i-1})y_3(t_{\iota-1}^{i-1}) \times \\
&\quad \left(\sum_{\iota=k-m_k^{i-1}}^{k-1} (u_j - y_j(t_{\iota-1}^{i-1}))y_1(t_{\iota-1}^{i-1}) \sum_{\iota=k-m_k^{i-1}}^{k-1} (u_j - y_j(t_{\iota-1}^{i-1}))y_j(t_{\iota-1}^{i-1})y_2(t_{\iota-1}^{i-1}) \right. \\
&\quad \left. - \sum_{\iota=k-m_k^{i-1}}^{k-1} (u_j - y_j(t_{\iota-1}^{i-1}))y_2(t_{\iota-1}^{i-1}) \sum_{\iota=k-m_k^{i-1}}^{k-1} (u_j - y_j(t_{\iota-1}^{i-1}))y_j(t_{\iota-1}^{i-1})y_1(t_{\iota-1}^{i-1}) \right) \\
&+ \sum_{\iota=k-m_k^{i-1}}^{k-1} (u_j - y_j(t_{\iota-1}^{i-1}))y_j^2(t_{\iota-1}^{i-1})y_1(t_{\iota-1}^{i-1}) \sum_{\iota=k-m_k^{i-1}}^{k-1} (u_j - y_j(t_{\iota-1}^{i-1}))y_j^3(t_{\iota-1}^{i-1})y_3(t_{\iota-1}^{i-1}) \times \\
&\quad \left(\sum_{\iota=k-m_k^{i-1}}^{k-1} (u_j - y_j(t_{\iota-1}^{i-1}))y_2(t_{\iota-1}^{i-1}) \sum_{\iota=k-m_k^{i-1}}^{k-1} \left[\sum_{\mathfrak{r}=1}^{n} \kappa_{j,\mathfrak{r}} y_{\mathfrak{r}}(t_{\iota-1}^{i-1})\right] y_j(t_{\iota-1}^{i-1}) \right. \\
&\quad \left. - \sum_{\iota=k-m_k^{i-1}}^{k-1} \left[\sum_{\mathfrak{r}=1}^{n} \kappa_{j,\mathfrak{r}} y_{\mathfrak{r}}(t_{\iota-1}^{i-1})\right] \sum_{\iota=k-m_k^{i-1}}^{k-1} (u_j - y_j(t_{\iota-1}^{i-1}))y_j(t_{\iota-1}^{i-1})y_2(t_{\iota-1}^{i-1}) \right)
\end{aligned}
$$

$$+ \sum_{\iota=k-m_k^{i-1}}^{k-1} (u_j - y_j(t_{\iota-1}^{i-1}))y_j^2(t_{\iota-1}^{i-1})y_2(t_{\iota-1}^{i-1}) \sum_{\iota=k-m_k^{i-1}}^{k-1} (u_j - y_j(t_{\iota-1}^{i-1}))y_j^3(t_{\iota-1}^{i-1})y_3(t_{\iota-1}^{i-1}) \times$$
$$\left(\sum_{\iota=k-m_k^{i-1}}^{k-1} \left[\sum_{\mathfrak{r}=1}^{n} \kappa_{j,\mathfrak{r}} y_{\mathfrak{r}}(t_{\iota-1}^{i-1})\right] \sum_{\iota=k-m_k^{i-1}}^{k-1} (u_j - y_j(t_{\iota-1}^{i-1}))y_j(t_{\iota-1}^{i-1})y_1(t_{\iota-1}^{i-1}) \right.$$
$$\left. - \sum_{\iota=k-m_k^{i-1}}^{k-1} (u_j - y_j(t_{\iota-1}^{i-1}))y_1(t_{\iota-1}^{i-1}) \sum_{\iota=k-m_k^{i-1}}^{k-1} \left[\sum_{\mathfrak{r}=1}^{n} \kappa_{j,\mathfrak{r}} y_{\mathfrak{r}}(t_{\iota-1}^{i-1})\right] y_j(t_{\iota-1}^{i-1}) \right).$$

For $j \neq l$, the determinant of the Jacobians $JF_2\left(\left\{\delta_{j,\mathfrak{r}}^{i-1}, \delta_{\mathfrak{r},\mathfrak{r}}^{i-1}\right\}_{\mathfrak{r}=1}^{3}\right)$, $JG_1\left(\beta_j^{i-1}, \left\{\gamma_{j,\mathfrak{r}}^{i-1}\right\}_{\mathfrak{r}=1}^{3}\right)$ and $JG_2\left(\left\{\sigma_{j,\mathfrak{r}}^{i-1}, \sigma_{\mathfrak{r},\mathfrak{r}}^{i-1}\right\}_{\mathfrak{r}=1}^{3}\right)$ can be derived in a similar way. For each $j \in I(1,3)$, it follows that the determinant of $JF_1\left(u_j^{i-1}, \left\{\kappa_{j,\mathfrak{r}}^{i-1}\right\}_{\mathfrak{r}=1}^{3}\right)$ is not zero provided that all parameters $\{\kappa_{j,\mathfrak{r}}\}_{\mathfrak{r}=1}^{3}$ are not zero or provided the sequence $\left\{y_j^{i-1}\left(t_{\mathfrak{r}}^{i-1}\right)\right\}_{\mathfrak{r}=k-m_k^{i-1}-1}^{k-1}$ is neither zero nor constant. To show this, suppose that the determinant of $JF_1\left(u_j^{i-1}, \left\{\kappa_{j,\mathfrak{r}}^{i-1}\right\}_{\mathfrak{r}=1}^{3}\right)$ is zero. This is equivalent to either one of the following;

- The rows of the matrix are dependent vectors in $\mathbb{R}^4$,
- The columns of the matrix are dependent vectors in $\mathbb{R}^4$.
- Either one of the rows or columns of the matrix is a zero vector.

This is equivalent to saying either all parameters $\{\kappa_{j,\mathfrak{r}}\}_{\mathfrak{r}=1}^{3}$ are zero, or the sequence $\left\{y_j^{i-1}\left(t_{\mathfrak{r}}^{i-1}\right)\right\}_{\mathfrak{r}=k-m_k^{i-1}-1}^{k-1}$ is zero or a constant.
Likewise, determinants of the Jacobians $JF_2\left(u_j^{i-1}, \left\{\delta_{j,\mathfrak{r}}^{i-1}, \delta_{\mathfrak{r},\mathfrak{r}}^{i-1}\right\}_{\mathfrak{r}=1}^{3}\right)$, $JG_1\left(\beta_j^{i-1}, \left\{\gamma_{j,\mathfrak{r}}^{i-1}\right\}_{\mathfrak{r}=1}^{3}\right)$ and $JG_2\left(\left\{\sigma_{j,\mathfrak{r}}^{i-1}, \sigma_{\mathfrak{r},\mathfrak{r}}\right\}_{\mathfrak{r}=1}^{3}\right)$ are non-zero if $\{\delta_{j,\mathfrak{r}}, \delta_{\mathfrak{r},\mathfrak{r}}\}_{\mathfrak{r}=1}^{3}$, $\{\gamma_{j,\mathfrak{r}}\}_{\mathfrak{r}=1}^{3}$ and $\{\sigma_{j,\mathfrak{r}}, \sigma_{\mathfrak{r},\mathfrak{r}}\}_{\mathfrak{r}=1}^{3}$ are not zero or provided the sequence $\left\{y_j^{i-1}\left(t_{\mathfrak{r}}^{i-1}\right)\right\}_{\mathfrak{r}=k-m_k^{i-1}-1}^{k-1}$, $\{y_{\mathfrak{r}}^{i-1}\left(t_{\mathfrak{r}}^{i-1}\right)\}_{\mathfrak{r}=k-m_k^{i-1}-1}^{k-1}$, $\left\{p_j^{i-1}\left(t_{\mathfrak{r}}^{i-1}\right)\right\}_{\mathfrak{r}=k-m_k^{i-1}-1}^{k-1}$ and $\{p_{\mathfrak{r}}^{i-1}\left(t_{\mathfrak{r}}^{i-1}\right)\}_{\mathfrak{r}=k-m_k^{i-1}-1}^{k-1}$ are neither zero nor constant for $j \neq l \in I(1,3)$.

REMARK 28 If the sample $\left\{y_j^{i-1}\left(t_{\mathfrak{r}}^{i-1}\right)\right\}_{\mathfrak{r}=k-m_k^{i-1}-1}^{k-1}$ is a constant sequence, it follows from (9.51) (q=1) and the fact that $\Delta\left(y_j^{i-1}(t_k^{i-1})\right) = 0$ and $s_{m_k^{i-1},k}^{j,j}(\Delta y_j) = 0$, that $u_j^{i-1}(m_k^{i-1}, t_k^{i-1}) \to \frac{1}{m_k^{i-1}} \sum_{\mathfrak{r}=k-m_k^{i-1}}^{k-1} y_j^{i-1}(t_{\mathfrak{r}}^{i-1})$. It also follows from (9.58) that $\{\kappa_{j,\mathfrak{r}}^{i-1}\}_{\mathfrak{r}=1}^{3}(m_k^{i-1}, t_k^{i-1}) \to 0$.

Chapter 10

Computational and Simulation Algorithms

10.1 Introduction

In this chapter, we outline computational, data organizational and simulation schemes. We introduce the ideas of iterative data process and data simulation time schedules in relation with the real time data observation/collection schedule. For the computational estimation of continuous time stochastic dynamic system state and parameters, it is essential to identify an admissible set of local conditional sample average and sample covariance parameters, namely, the size of local conditional sample in the context of a partition of time interval $[T_{i-1} - \tau_{i-1}, T_i]$. Moreover, the discrete time dynamic model of conditional sample mean and sample covariance statistic processes in Section 9.5 and the theoretical parameter estimation scheme in Section 9.6 motivates to outline a computational scheme in a systematic and coherent manner. A brief conceptual computational scheme and simulation process summary is described below:

10.2 Coordination of Data Observation, Iterative Process, and Simulation Schedules:

Without loss of generality, we assume that the real data observation/collection partition schedules $\mathbb{P}^{i-1}$, $i \in I(1, K^*)$ are defined in (9.29). Now, we present definitions of iterative process and simulation time schedule.

DEFINITION 10.2.1 *The iterative process time schedule in relation with the real data collection schedule is defined by*

$$\Big\{ \; I\mathbb{P}^{i-1} \;\; = \;\; \{F^{-r_{i-1}} t_k^{i-1} : \; \textit{for} \;\; t_k^{i-1} \in \mathbb{P}^{i-1}\}, \;\; \textit{for}\; i \in I(1, K^*), \;\; k \in I(-r_{i-1}, N_{i-1}), \tag{10.1}$$

where $F^{-r_{i-1}} t_k^{i-1} = t_{k-r_{i-1}}^{i-1}$ *is a forward shift operator [11].*

The simulation time is based on the order d_{i-1} *of the time series model of* m_k^{i-1}*-local conditional sample mean and covariance processes in Lemma* 9.3.

REMARK 29 For the case where $K = 0$, we have $I\mathbb{P}_{i-1} = I\mathbb{P}$, where $\mathbb{P}^{i-1} = \mathbb{P}$ is defined in (9.28). This is the iterative time schedule in the absence of jumps.

DEFINITION 10.2.2 *The simulation process time schedule in relation with the real data observation schedule is defined by*

$$S\mathbb{P}^{i-1} = \begin{cases} \{F^{r_{i-1}} t_k^{i-1} : \ \textit{for} \ \ t_k^{i-1} \in \mathbb{P}^{i-1}\}, \ \ \textit{if} \ \ d_{i-1} \leq r_{i-1} \\ \{F^{p_{i-1}} t_k^{i-1} : \ \textit{for} \ \ t_k^{i-1} \in \mathbb{P}^{i-1}\}, \ \ \textit{if} \ \ d_{i-1} > r_{i-1}, \ \ k \in I(-r_{i-1}, N_{i-1}). \end{cases} \tag{10.2}$$

REMARK 30 For each $i \in I(1, K^*)$, the initial times of iterative and simulation processes are equal to the real data times $t_{r_{i-1}}^{i-1}$ and $t_{d_{i-1}}^{i-1}$, whenever $d_{i-1} \leq r_{i-1}$ and $d_{i-1} > r_{i-1}$, respectively. The iterative process and simulation process times with jump are $t_{k+r_{i-1}}^{i-1}$ and $t_{k+d_{i-1}}^{i-1}$, $i \in I(1, K^*)$, respectively.

10.3 Conceptual Computational Parameter Estimation Scheme

For the conceptual computational dynamic system parameter estimation, we need to introduce a few concepts of local admissible sample/data observation size m_k^{i-1}-local admissible conditional finite sequence at $t_k^{i-1} \in S\mathbb{P}^{i-1}$, local finite sequence of parameter estimates at t_k^{i-1}.

DEFINITION 10.3.1 *For each $i \in I(1, K^*)$, and $t_k^{i-1} \in I(T_{i-1} - \tau_{i-1}, T_i)$, we define local admissible sample/data observation size m_k^{i-1} at t_k^{i-1} as $m_k^{i-1} \in OS_k^{i-1}$, where*

$$OS_k^{i-1} = \begin{cases} I(2, r_{i-1} + \mathcal{S}_{i-1} + k - 1), \ \ \textit{if} \, d_{i-1} \leq r_{i-1}, \\ I(2, d_{i-1} + \mathcal{S}_{i-1} + k - 1), \ \ \textit{if} \, d_{i-1} > r_{i-1}, \ k \in I(0, N_{i-1}) \end{cases} \tag{10.3}$$

Moreover, OS_k^{i-1} is referred as the local admissible set of lagged sample/data observation size at t_k^{i-1}.

REMARK 31 We note that if $K = 0$, $\mathcal{S}_{i-1} = 0$, the point $t_k^{i-1} = t_k \in [t_0, T]$. Thus, (10.3) reduces to

$$OS_k^{i-1} = \begin{cases} I(2, r + k - 1), \ \ \text{if} \ d \leq r, \\ I(2, d + k - 1), \ \ \text{if} \ d > r, \ k \in I(0, N) \end{cases}$$

DEFINITION 10.3.2 *For each $i \in I(1, K^*)$, $m_k^{i-1} \in OS_k^{i-1}$ in Definition 10.3.1 and $k \in I(0, N_{i-1})$, a m_k^{i-1}-local admissible lagged-adapted finite restriction sequence of conditional sample/data observation at time t_k^{i-1} to subpartition P_k^{i-1} of $\mathbb{P}^{i-1}$ in Definition 9.5.2 is defined by*

$\left(\{\mathbb{E}[\boldsymbol{y}^{i-1}(t_l^{i-1})|\mathcal{F}_{l-1}^{i-1}]\}_{l=k-m_k^{i-1}}^{k-1}, \{\mathbb{E}[\boldsymbol{p}^{i-1}(t_l^{i-1})|\mathcal{F}_{l-1}^{i-1}]\}_{l=k-m_k^{i-1}}^{k-1}\right)$. Moreover, a m_k^{i-1}- class of admissible lagged-adapted finite sequences of conditional sample/data observation of size m_k^{i-1} at t_k^{i-1} is defined by

$$\mathcal{AS}_k^{i-1} = \begin{cases} \left\{\{\mathbb{E}[\boldsymbol{y}^{i-1}(t_l^{i-1})|\mathcal{F}_{l-1}^{i-1}]\}_{l=k-m_k^{i-1}}^{k-1}\right\}_{m_k^{i-1}\in OS_k^{i-1}}, \\ \left\{\{\mathbb{E}[\boldsymbol{p}^{i-1}(t_l^{i-1})|\mathcal{F}_{l-1}^{i-1}]\}_{l=k-m_k^{i-1}}^{k-1}\right\}_{m_k^{i-1}\in OS_k^{i-1}}. \end{cases} \tag{10.4}$$

In the case of energy commodity model, for each $i \in I(1, K^*)$, $m_k^{i-1} \in OS_k^{i-1}$, we find corresponding m_k^{i-1}- local admissible adapted finite sequence of conditional sample/data observation at t_k^{i-1}, $\left(\{\mathbb{E}[\boldsymbol{y}^{i-1}(t_l^{i-1})|\mathcal{F}_{l-1}^{i-1}]\}_{l=k-m_k^{i-1}}^{k-1}, \{\mathbb{E}[\boldsymbol{p}^{i-1}(t_l^{i-1})|\mathcal{F}_{l-1}^{i-1}]\}_{l=k-m_k^{i-1}}^{k-1}\right)$. For $i \in I(1, K^*)$, using this sequence and solutions of (9.63), we compute

$$\begin{cases} u_j^{i-1}(m_k, t_k^{i-1}), \beta_j^{i-1}(m_k^{i-1}, t_k^{i-1}), \kappa_{j,l}^{i-1}(m_k^{i-1}, t_k^{i-1}), \gamma_{j,l}^{i-1}(m_k^{i-1}, t_k^{i-1}), \delta_{j,l}^{i-1}(m_k^{i-1}, t_k^{i-1}), \\ \sigma_{j,l}^{i-1}(m_k^{i-1}, t_k^{i-1}),\ k \in [0, N_{i-1}],\ \textit{for}\ j, l \in I(1, n). \end{cases}$$

This leads to a local finite sequence of parameter estimates at t_k^{i-1} defined on OS_k^{i-1} as follows:

$$\Big\{\hat{u}_j^{i-1}(m_k^{i-1}, t_k^{i-1}), \hat{\beta}_j^{i-1}(m_k^{i-1}, t_k^{i-1}), \hat{\kappa}_{j,l}^{i-1}(m_k^{i-1}, t_k^{i-1}), \hat{\gamma}_{j,l}^{i-1}(m_k^{i-1}, t_k^{i-1}), \hat{\delta}_{j,l}^{i-1}(m_k^{i-1}, t_k^{i-1}),$$
$$\hat{\sigma}_{j,l}^{i-1}(m_k^{i-1}, t_k^{i-1})\Big\}_{m_k^{i-1}\in OS_k^{i-1}}.$$

The above defined collection is denoted by

$$(\mathcal{U}_k, \mathcal{B}_k, \mathcal{K}_k, \boldsymbol{\gamma}_k, \boldsymbol{\delta}_k, \boldsymbol{\sigma}_k) = \begin{cases} \Big\{\hat{u}_j^{i-1}(m_k^{i-1}, t_k^{i-1}), \hat{\beta}_j^{i-1}(m_k^{i-1}, t_k^{i-1}), \hat{\kappa}_{j,l}^{i-1}(m_k^{i-1}, t_k^{i-1}), \\ \hat{\gamma}_{j,l}^{i-1}(m_k^{i-1}, t_k^{i-1})\hat{\delta}_{j,l}^{i-1}(m_k^{i-1}, t_k^{i-1}), \hat{\sigma}_{j,l}^{i-1}(m_k^{i-1}, t_k^{i-1})\Big\}_{m_k^{i-1}\in OS_k^{i-1}}, \\ \textit{for}\ j \in I(1, n),\ i \in I(1, K^*). \end{cases}$$

10.4 Conceptual Computation of State Simulation Scheme

For the development of a conceptual computational scheme, we need to employ the method of induction. The presented simulation scheme is based on the idea of lagged adaptive expectation process [88]. For $j, l \in I(1, n)$, an autocorrelation function (ACF) analysis [14, 11] performed on $\left(s_{m_k^{i-1},k}^{j,j}(y), s_{m_k^{i-1},k}^{j,j}(p)\right)$ suggests that the interconnected discrete time dynamic model of local conditional sample mean and sample variance statistics in Lemma 9.3 is of order $d_{i-1} = 2$. In view of this, we need to identify the initial data. We begin with a given initial data $\left(\mathbf{y}^{i-1}(T_{i-1}), \mathbf{p}^{i-1}(T_{i-1})\right)$, $\left(\{\Sigma_{m_0^{i-1}, t_0^{i-1}}(y)\}_{m_0^{i-1}\in OS_0^{i-1}}, \{\Sigma_{m_0^{i-1}, t_0^{i-1}}(p)\}_{m_0^{i-1}\in OS_0^{i-1}}\right)$,

$\left(\{\Sigma_{m_{-1}^{i-1},t_{-1}^{i-1}}(y)\}_{m_{-1}^{i-1}\in OS_{-1}^{i-1}}, \{\Sigma_{m_{-1}^{i-1},t_{-1}^{i-1}}(p)\}_{m_{-1}^{i-1}\in OS_{-1}^{i-1}}\right)$,
$\left(\{\bar{S}_{m_{-1}^{i-1},t_{-1}^{i-1}}(y)\}_{m_{-1}^{i-1}\in OS_{-1}^{i-1}}, \{\bar{S}_{m_{-1}^{i-1},t_{-1}^{i-1}}(p)\}_{m_{-1}^{i-1}\in OS_{-1}^{i-1}}\right)$. Let $\left(\mathbf{y}^s(m_k^{i-1}, t_k^{i-1}), \mathbf{p}^s(m_k^{i-1}, t_k^{i-1})\right)$ be a simulated value of $\left(\mathbb{E}[\mathbf{y}^{i-1}(t_k^{i-1})|\mathcal{F}_{k-1}^{i-1}], \mathbb{E}[\mathbf{p}^{i-1}(t_k^{i-1})|\mathcal{F}_{k-1}^{i-1}]\right)$ at time t_k^{i-1} corresponding to an admissible sequence $\{\mathbb{E}[\mathbf{y}^{i-1}(t_l^{i-1})|\mathcal{F}_{l-1}^{i-1}], \mathbb{E}[\mathbf{p}^{i-1}(t_l^{i-1})|\mathcal{F}_{l-1}^{i-1}]\}_{l=k-m_k}^{k-1} \in \mathcal{AS}_k^{i-1}$. For $q = 1$, and $j \in I(1, n)$, the simulated value
$\left(y_j^s(m_k^{i-1}, t_k^{i-1}) \equiv y_j^{i-1,s}(m_k^{i-1}, t_k^{i-1}), p_j^s(m_k^{i-1}, t_k^{i-1}) \equiv p_j^{i-1,s}(m_k^{i-1}, t_k^{i-1})\right)$ is generated from the discretized Euler scheme (9.51)-(9.52) as follows:

$$\begin{cases}
y_j^s(m_k^{i-1}, t_k^{i-1}) &= y_j^s(m_{k-1}^{i-1}, t_{k-1}^{i-1}) + \left(u_j^{i-1}(m_{k-1}^{i-1}, t_{k-1}^{i-1}) - y_j^s(m_{k-1}^{i-1}, t_{k-1}^{i-1})\right) \times \\
&\quad \left[\sum_{l=1}^n \kappa_{j,l}^{i-1}(m_{k-1}^{i-1}, t_{k-1}^{i-1}) y_l^s(m_{k-1}^{i-1}, t_{k-1}^{i-1})\right] \Delta t + \left(u_j^{i-1}(m_{k-1}^{i-1}, t_{k-1}^{i-1})\right. \\
&\quad \left. - y_j^s(m_{k-1}^{i-1}, t_{k-1}^{i-1})\right) \left[\delta_{j,j}^{i-1}(m_{k-1}^{i-1}, t_{k-1}^{i-1}) \Delta W_{j,j}(m_k^{i-1}, t_k^{i-1})\right. \\
&\quad \left. + \sum_{l\neq j}^n \delta_{j,l}^{i-1}(m_{k-1}^{i-1}, t_{k-1}^{i-1}) y_l^s(m_{k-1}^{i-1}, t_{k-1}^{i-1}) \Delta W_{j,l}(m_k^{i-1}, t_k^{i-1})\right], \\
&\quad t_k^{i-1} \in [T_{i-1}, T_i), \\
y_j^{i,s}(T_i^-) &= \pi_j^i y_j^{i-1,s}\left(T_i^-, T_{i-1}, \mathbf{y}^{i-1,s}\right), \\
p_j^s(m_k^{i-1}, t_k^{i-1}) &= p_j^s(m_{k-1}^{i-1}, t_{k-1}^{i-1}) \left(1 + \left[\gamma_{j,j}^{i-1}(m_{k-1}^{i-1}, t_{k-1}^{i-1}) \left(y_j^s(m_{k-1}^{i-1}, t_{k-1}^{i-1})\right.\right.\right. \\
&\quad \left. - p_j^s(m_{k-1}^{i-1}, t_{k-1}^{i-1})\right) \\
&\quad \left.\left. + \beta_j^{i-1}(m_{k-1}^{i-1}, t_{k-1}^{i-1}) + \sum_{l\neq j}^n \gamma_{j,l}^{i-1}(m_{k-1}^{i-1}, t_{k-1}^{i-1}) p_l^s(m_{k-1}^{i-1}, t_{k-1}^{i-1})\right] \Delta t\right) \\
&\quad + \sigma_{j,j}^{i-1}(m_{k-1}^{i-1}, t_{k-1}^{i-1}) p_j^s(m_{k-1}^{i-1}, t_{k-1}^{i-1}) \Delta Z_{j,j}(m_k^{i-1}, t_k^{i-1}) \\
&\quad + p_j^s(m_{k-1}^{i-1}, t_{k-1}^{i-1}) \sum_{l\neq j}^n \sigma_{j,l}^{i-1}(m_{k-1}^{i-1}, t_{k-1}^{i-1}) p_l^s(m_{k-1}^{i-1}, t_{k-1}^{i-1}) \Delta Z_{j,l}(m_k^{i-1}, t_k^{i-1}), \\
&\quad t_k^{i-1} \in [T_{i-1}, T_i), \\
p_j^{i,s}(T_i^-) &= \theta_j^i p_j^{i-1,s}\left(T_i^-, T_{i-1}, \mathbf{y}^{i-1,s}, \mathbf{p}^{i-1,s}\right).
\end{cases} \tag{10.5}$$

To find the simulated value $y_j^{i,s}(T_i)$ and $p_j^{i,s}(T_i)$, we need to estimate $\hat{\pi}_j^i$ and $\hat{\theta}_j^i$ by first simulating

$$\lim_{t\to T_i^-} y_j^{i-1}\left(t, T_{i-1}, \mathbf{y}^{i-1,s}\right) \equiv y_j^s(m_{N_{i-1}}^{i-1}, t_{N_{i-1}}^{i-1})$$

and

$$\lim_{t\to T_i^-} p_j^{i-1}\left(t, T_{i-1}, \mathbf{y}^{i-1,s}, \mathbf{p}^{i-1,s}\right) \equiv p_j^s(m_{N_{i-1}}^{i-1}, t_{N_{i-1}}^{i-1})$$

as follows:

$$
\begin{aligned}
y_j^s(m_{N_{i-1}}^{i-1}, t_{N_{i-1}}^{i-1}) &= y_j^s(m_{N_{i-1}-1}^{i-1}, t_{N_{i-1}-1}^{i-1}) \\
&+ \Big(u_j^{i-1}(m_{N_{i-1}-1}^{i-1}, t_{N_{i-1}-1}^{i-1}) - y_j^s(m_{N_{i-1}-1}^{i-1}, t_{N_{i-1}-1}^{i-1})\Big) \times \\
&\Bigg[\sum_{l=1}^{n} \kappa_{j,l}^{i-1}(m_{N_{i-1}-1}^{i-1}, t_{N_{i-1}-1}^{i-1}) y_l^s(m_{N_{i-1}-1}^{i-1}, t_{N_{i-1}-1}^{i-1})\Delta t \\
&+ \delta_{j,j}^{i-1}(m_{N_{i-1}-1}^{i-1}, t_{N_{i-1}-1}^{i-1})\Delta W_{j,j}(m_{N_{i-1}}^{i-1}, t_{N_{i-1}}^{i-1}) \\
&+ \sum_{l\neq j}^{n} \delta_{j,l}^{i-1}(m_{N_{i-1}-1}^{i-1}, t_{N_{i-1}-1}^{i-1}) y_l^s(m_{N_{i-1}-1}^{i-1}, t_{N_{i-1}-1}^{i-1})\Delta W_{j,l}(m_{N_{i-1}}^{i-1}, t_{N_{i-1}}^{i-1})\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
p_j^s(m_{N_{i-1}}^{i-1}, t_{N_{i-1}}^{i-1}) &= (p_j^s(m_{N_{i-1}-1}^{i-1}, t_{N_{i-1}-1}^{i-1}) \\
&+ p_j^s(m_{N_{i-1}-1}^{i-1}, t_{N_{i-1}-1}^{i-1})\Big[\gamma_{j,j}^{i-1}(m_{N_{i-1}-1}^{i-1}, t_{N_{i-1}-1}^{i-1})\Big(y_j^s(m_{N_{i-1}-1}^{i-1}, t_{N_{i-1}-1}^{i-1}) \\
&- p_j^s(m_{N_{i-1}-1}^{i-1}, t_{N_{i-1}-1}^{i-1})\Big) + \beta_j^{i-1}(m_{N_{i-1}-1}^{i-1}, t_{N_{i-1}-1}^{i-1}) \\
&+ \sum_{l\neq j}^{n} \gamma_{j,l}^{i-1}(m_{N_{i-1}-1}^{i-1}, t_{N_{i-1}-1}^{i-1}) p_l^s(m_{N_{i-1}-1}^{i-1}, t_{N_{i-1}-1}^{i-1})\Big]\Delta t \\
&+ p_j^s(m_{N_{i-1}-1}^{i-1}, t_{N_{i-1}-1}^{i-1})\Big[\sigma_{j,j}^{i-1}(m_{N_{i-1}-1}^{i-1}, t_{N_{i-1}-1}^{i-1})\Delta Z_{j,j}(m_{N_{i-1}}^{i-1}, t_{N_{i-1}}^{i-1}) \\
&+ \sum_{l\neq j}^{n} \sigma_{j,l}^{i-1}(m_{N_{i-1}-1}^{i-1}, t_{N_{i-1}-1}^{i-1}) p_l^s(m_{N_{i-1}-1}^{i-1}, t_{N_{i-1}-1}^{i-1})\Delta Z_{j,l}(m_{N_{i-1}}^{i-1}, t_{N_{i-1}}^{i-1})\Big]
\end{aligned}
$$

From this, we calculate $\hat{\pi}_j^i$ and $\hat{\theta}_j^i$ as:

$$
\begin{aligned}
\hat{\pi}_j^i &= \frac{\mathbb{E}[y_j^{i-1}(T_i)|\mathcal{F}_{T_i-1}^{i-1}]}{y_j^s(m_{N_{i-1}}^{i-1}, t_{N_{i-1}}^{i-1})} \\
\hat{\theta}_j^i &= \frac{\mathbb{E}[p_j^{i-1}(T_i)|\mathcal{F}_{T_i-1}^{i-1}]}{p_j^s(m_{N_{i-1}}^{i-1}, t_{N_{i-1}}^{i-1})}.
\end{aligned} \tag{10.6}
$$

Thus, $y_j^{i,s}(T_i) = \hat{\pi}_j^i y_j^{i-1,s}\left(T_i^-, T_{i-1}, \mathbf{y}^{i-1,s}\right)$ and $p_j^{i,s}(T_i) = \hat{\theta}_j^i p_j^{i-1,s}\left(T_i^-, T_{i-1}, \mathbf{y}^{i-1,s}, \mathbf{p}^{i-1,s}\right)$.

Let $(\{\mathbf{y}^s(m_k^{i-1}, t_k^{i-1})\}_{m_k^{i-1}\in OS_k^{i-1}}, \{\mathbf{p}^s(m_k^{i-1}, t_k^{i-1})\}_{m_k^{i-1}\in OS_k^{i-1}})$ be a m_k^{i-1}- local sequence of simulated values corresponding to m_k^{i-1}-admissible lagged adapted finite sequence of conditional observation belonging to $\mathcal{AS}_k^{i-1}$, and corresponding term of sequence $(\mathcal{U}_k, \mathcal{B}_k, \mathcal{K}_k, \boldsymbol{\gamma}_k, \boldsymbol{\delta}_k, \boldsymbol{\sigma}_k)$. Thus, for each $i \in I(1, K^*)$, $(\{\mathbf{y}^s(m_k^{i-1}, t_k^{i-1})\}_{m_k^{i-1}\in OS_k^{i-1}}, \{\mathbf{p}^s(m_k^{i-1}, t_k^{i-1})\}_{m_k^{i-1}\in OS_k^{i-1}})$ are the finite sequence correspondence of simulated values of $(\mathbb{E}[\mathbf{y}^{i-1}(t_k^{i-1})|\mathcal{F}_{k-1}^{i-1}], \mathbb{E}[\mathbf{p}^{i-1}(t_k^{i-1})|\mathcal{F}_{k-1}^{i-1}])$ at t_k^{i-1}.

10.5 Mean-Square Sub-Optimal Procedure

To find the best estimate of $(\mathbb{E}[\mathbf{y}(t_k^{i-1})|\mathcal{F}_{k-1}^{i-1}], \mathbb{E}[\mathbf{p}(t_k^{i-1})|\mathcal{F}_{k-1}^{i-1}])$ using a local admissible finite sequence $(\{\mathbf{y}^s(m_k^{i-1}, t_k^{i-1})\}_{m_k^{i-1}\in OS_k^{i-1}}, \{\mathbf{p}^s(m_k^{i-1}, t_k^{i-1})\}_{m_k^{i-1}\in OS_k^{i-1}})$, we need to compute a finite sequence of quadratic mean square error corresponding to $(\{\mathbf{y}^s(m_k^{i-1}, t_k^{i-1})\}_{m_k^{i-1}\in OS_k^{i-1}}, \{\mathbf{p}^s(m_k^{i-1}, t_k^{i-1})\}_{m_k^{i-1}\in OS_k^{i-1}})$. The quadratic mean square error is defined below.

DEFINITION 10.5.1 *For each $i \in I(1, K^*)$, the quadratic mean square error of $(\mathbb{E}[\boldsymbol{y}(t_k^{i-1})|\mathcal{F}_{k-1}^{i-1}], \mathbb{E}[\boldsymbol{p}(t_k^{i-1})|\mathcal{F}_{k-1}^{i-1}])$ relative to each member of the term of local admissible sequence $(\{\mathbf{y}^s(m_k^{i-1}, t_k^{i-1})\}_{m_k^{i-1}\in OS_k^{i-1}}, \{\boldsymbol{p}^s(m_k^{i-1}, t_k^{i-1})\}_{m_k^{i-1}\in OS_k^{i-1}})$ of simulated values is defined by*

$$\Xi_{m_k^{i-1}, t_k^{i-1}} = \left\|\boldsymbol{y}^s(m_k^{i-1}, t_k^{i-1}) - \mathbb{E}[\boldsymbol{y}^{i-1}(t_k^{i-1})|\mathcal{F}_{k-1}^{i-1}]\right\|^2 + \left\|\boldsymbol{p}^s(m_k^{i-1}, t_k^{i-1}) - \mathbb{E}[\boldsymbol{p}^{i-1}(t_k^{i-1})|\mathcal{F}_{k-1}^{i-1}]\right\|^2. \tag{10.7}$$

For any arbitrary small positive number ϵ and for each time t_k^{i-1} to find the the best estimate from the admissible simulated values of simulated sequence of $\{\mathbf{y}^s(m_k^{i-1}, t_k^{i-1})\}_{m_k^{i-1}\in OS_k^{i-1}}, \{\mathbf{p}^s(m_k^{i-1}, t_k^{i-1})\}_{m_k^{i-1}\in OS_k^{i-1}}$ for $\mathbb{E}[\mathbf{y}(t_k^{i-1})|\mathcal{F}_{k-1}^{i-1}], \mathbb{E}[\mathbf{p}(t_k^{i-1})|\mathcal{F}_{k-1}^{i-1}]$, we determine the following sub-optimal admissible set of m_k^{i-1}-size local conditional sample

$$\left\{ \mathcal{M}_{t_k^{i-1}} = \{m_k^{i-1} \in OS_k^{i-1} \ : \ \Xi_{m_k^{i-1}, t_k^{i-1}} < \epsilon\}, \text{ for } i \in I(1, K^*). \right. \tag{10.8}$$

Among these collected values, the value that gives the minimum $\Xi_{m_k^{i-1}, t_k^{i-1}}$ for $k \in [0, N_{i-1}]$ are recorded as $\hat{m}_k^{i-1}$. If more than one value exist, then the largest of such m_k^{i-1}'s is recorded as $\hat{m}_k^{i-1}$. If condition (10.8) is not met at time t_k^{i-1}, the value of m_k^{i-1} where the minimum $\min\limits_{m_k^{i-1}} \Xi_{m_k^{i-1}, t_k^{i-1}}$ is attained is recorded as $\hat{m}_k^{i-1}$. The $\epsilon-$ level sub-optimal estimates of the parameters $\{\ \hat{u}_j^{i-1}(m_k^{i-1}, t_k^{i-1}), \hat{\kappa}_{j,l}^{i-1}(m_k^{i-1}, t_k^{i-1}), \hat{\beta}_j^{i-1}(m_k^{i-1}, t_k^{i-1}), \hat{\delta}_{j,l}^{i-1}(m_k^{i-1}, t_k^{i-1})\ , \hat{\gamma}_{j,l}^{i-1}(m_k^{i-1}, t_k^{i-1}), \hat{\sigma}_{j,l}^{i-1}(m_k^{i-1}, t_k^{i-1})\ \}$ are recorded as $\{\ u_j^{i-1}(\hat{m}_{k,y_j}^{i-1}, k),\ \kappa_{j,l}^{i-1}(\hat{m}_{k,y_j}^{i-1}, k),\ \beta_j^{i-1}(\hat{m}_{k,p_j}^{i-1}, k),\ \delta_{j,l}^{i-1}(\hat{m}_{k,y_j}^{i-1}, k),\ \gamma_{j,l}^{i-1}(\hat{m}_{k,p_j}^{i-1}, k),\ \sigma_{j,l}^{i-1}(\hat{m}_{k,p_j}^{i-1}, k)\ \}$. Finally, the simulated value $\mathbf{y}^s(m_k^{i-1}, t_k^{i-1})$, $\mathbf{p}^s(m_k^{i-1}, t_k^{i-1})$ at time t_k^{i-1} with $\hat{m}_k^{i-1}$ is now recorded as the best estimate for $\mathbb{E}[\mathbf{y}^{i-1}(t_k^{i-1})|\mathcal{F}_{k-1}^{i-1}]$ and $\mathbb{E}[\mathbf{p}^{i-1}(t_k^{i-1})|\mathcal{F}_{k-1}^{i-1}]$. The value $\mathbf{y}^s(\hat{m}_k^{i-1}, k)$, $\mathbf{p}^s(\hat{m}_k^{i-1}, k)$ is called the $\epsilon-$ sub-optimal simulated value of $\mathbf{y}^s(m_k^{i-1}, t_k^{i-1})$ and $\mathbf{p}^s(m_k^{i-1}, t_k^{i-1})$ of $\mathbb{E}[\mathbf{y}^{i-1}(t_k^{i-1})|\mathcal{F}_{k-1}^{i-1}]$ and $\mathbb{E}[\mathbf{p}^{i-1}(t_k^{i-1})|\mathcal{F}_{k-1}^{i-1}]$ at t_k^{i-1}.

10.6 Illustration: Application of Conceptual Computational Algorithm to Energy Commodity Data Set

In this subsection, we apply the above conceptual computational algorithm to study the relationship between three energy commodities by setting $n = 3$ in (9.45). The three energy commodities are daily Henry Hub Natural gas data set, daily crude oil data set, and daily coal data set for the period of $05/04/2009 - 01/03/2014$, [26, 27, 28]. Thus, for each pair (y_1, p_1), (y_2, p_2), and (y_3, p_3), the drift and diffusion coefficient function of the stochastic dynamic equation governing (y_j, p_j), for $j \in I(1, 3)$ have 4 and 3 parameters each to be estimated, respectively. Thus, there are 42 parameters to be estimated in total. Using $\Delta t = 1$, $\epsilon = 0.001$, for each $j \in I(1, 3)$, the $\epsilon-$ level sub-optimal estimates of parameters $u_j^{i-1}(\hat{m}_k^{i-1}, k)$, $\beta_j^{i-1}(\hat{m}_k^{i-1}, k)$, $\kappa_{j,l}^{i-1}(\hat{m}_k^{i-1}, k)$, $\gamma_{j,l}^{i-1}(\hat{m}_k^{i-1}, k)$, $\delta_{j,l}^{i-1}(\hat{m}_k^{i-1}, k)$, $\sigma_{j,l}^{i-1}(\hat{m}_k^{i-1}, k)$, $l \in I(1, 3)$, at each real data times are exhibited below.

10.6.1 Illustration: Relationship between Natural Gas, Crude Oil and Coal: Without Incorporating Jump Process.

In this subsubsection, we analyze the relationship between Natural Gas, Crude Oil, and Coal without the jump process. For $j, l \in I(1, 3)$, the stochastic dynamic system governing the three energy commodities is described in (9.48) of Remark (27). Here, (y_1, p_1) denotes the mean spot and the spot price process of Natural gas, (y_2, p_2) denotes the mean spot and the spot price process of Crude oil, and (y_3, p_3) denotes the mean spot and the spot price process of Coal.

Using the discretized scheme (10.5), we apply the above conceptual computational algorithm for the real time data sets namely daily Henry Hub Natural gas data set, daily crude oil data set, and daily coal data set. Using $r = 10$, and $d = 2$, the $\epsilon-$ level sub-optimal estimates of the parameters at each real data times are described below.

The parameters corresponding to the natural gas data set are $u_1(\hat{m}_k, k)$, $\beta_1(\hat{m}_k, k)$, $\kappa_{1,1}(\hat{m}_k, k)$, $\kappa_{1,2}(\hat{m}_k, k)$, $\kappa_{1,3}(\hat{m}_k, k)$, $\gamma_{1,1}(\hat{m}_k, k)$, $\gamma_{1,2}(\hat{m}_k, k)$, $\gamma_{1,3}(\hat{m}_k, k)$, $\delta_{1,1}(\hat{m}_k, k)$, $\delta_{1,2}(\hat{m}_k, k)$, $\delta_{1,3}(\hat{m}_k, k)$, $\sigma_{1,1}(\hat{m}_k, k)$, $\sigma_{1,2}(\hat{m}_k, k)$, $\sigma_{1,3}(\hat{m}_k, k)$. The parameters corresponding to the crude oil data set are $u_2(\hat{m}_k, k)$, $\beta_2(\hat{m}_k, k)$, $\kappa_{2,1}(\hat{m}_k, k)$, $\kappa_{2,2}(\hat{m}_k, k)$, $\kappa_{2,3}(\hat{m}_k, k)$, $\gamma_{2,1}(\hat{m}_k, k)$, $\gamma_{2,2}(\hat{m}_k, k)$, $\gamma_{2,3}(\hat{m}_k, k)$, $\delta_{2,1}(\hat{m}_k, k)$, $\delta_{2,2}(\hat{m}_k, k)$, $\delta_{2,3}(\hat{m}_k, k)$, $\sigma_{2,1}(\hat{m}_k, k)$, $\sigma_{2,2}(\hat{m}_k, k)$, $\sigma_{2,3}(\hat{m}_k, k)$. The parameters corresponding to coal data set are $u_3(\hat{m}_k, k)$, $\beta_3(\hat{m}_{k,p_3}, k)$, $\kappa_{3,1}(\hat{m}_k, k)$, $\kappa_{3,2}(\hat{m}_k, k)$, $\kappa_{3,3}(\hat{m}_k, k)$, $\gamma_{3,1}(\hat{m}_k, k)$, $\gamma_{3,2}(\hat{m}_k, k)$, $\gamma_{3,3}(\hat{m}_k, k)$, $\delta_{3,1}(\hat{m}_k, k)$, $\delta_{3,2}(\hat{m}_k, k)$, $\delta_{3,3}(\hat{m}_k, k)$, $\sigma_{3,1}(\hat{m}_k, k)$, $\sigma_{3,2}(\hat{m}_k, k)$, $\sigma_{3,3}(\hat{m}_k, k)$.

The following table gives the parameter estimates $u_1(\hat{m}_k, k)$, $\kappa_{1,1}(\hat{m}_k, k)$, $\kappa_{1,2}(\hat{m}_k, k)$, $\kappa_{1,3}(\hat{m}_k, k)$, u_2, $\kappa_{2,1}(\hat{m}_k, k)$, $\kappa_{2,2}(\hat{m}_k, k)$, $\kappa_{2,3}(\hat{m}_k, k)$, $u_3(\hat{m}_k, k)$, $\kappa_{3,1}(\hat{m}_k, k)$, $\kappa_{3,2}(\hat{m}_k, k)$, $\kappa_{3,3}(\hat{m}_k, k)$ for the decoupled system for **y** in the case where jump is not incorporated into the system.

Table 15: Estimates $\hat{m}_k$, $u_1(\hat{m}_k, k)$, $\kappa_{1,1}(\hat{m}_k, k)$, $\kappa_{1,2}(\hat{m}_k, k)$, $\kappa_{1,3}(\hat{m}_k, k)$, $u_2(\hat{m}_k, k)$, $\kappa_{2,1}(\hat{m}_k, k)$, $\kappa_{2,2}(\hat{m}_k, k)$, $\kappa_{2,3}(\hat{m}_k, k)$, $u_3(\hat{m}_k, k)$, $\kappa_{3,1}(\hat{m}_k, k)$, $\kappa_{3,2}(\hat{m}_k, k)$, $\kappa_{3,3}(\hat{m}_k, k)$ (without jump).

t_k		Natural gas				Crude oil				Coal			
	$\hat{m}_k$	u_1	$\kappa_{1,1}$	$\kappa_{1,2}$	$\kappa_{1,3}$	u_2	$\kappa_{2,1}$	$\kappa_{2,2}$	$\kappa_{2,3}$	u_3	$\kappa_{3,1}$	$\kappa_{3,2}$	$\kappa_{3,3}$
				$\times 10^{-16}$	$\times 10^{-18}$		$\times 10^{-18}$		$\times 10^{-18}$		$\times 10^{-18}$	$\times 10^{-18}$	
11	1	4.1593	0.0211	0	0	57.7000	0	0	0	16.7407	0	0	0
12	3	4.2000	0.0111	0	0	58.6313	0.0011	0.0310	-0.0012	16.2395	0	0	-0.0376
13	5	4.0616	0.0679	-0.0054	-0.0035	58.5378	-0.0035	0.0205	0.0032	16.2680	0	0	0.1069
14	5	4.0616	-0.0242	-0.0179	0	61.4809	0.0020	0.0098	0	15.5249	0	0	-0.0294
15	8	4.0910	0.6416	-0.2898	0	58.9282	-0.0036	0.0128	0.0071	16.8286	0	0	0.0513
16	8	4.0160	0.2101	0	0	59.6867	-0.0051	0.0080	0.0071	17.0888	0	0	0.0415
17	8	4.9575	0.1876	0	0	60.6244	0.0024	0.0052	0	17.4120	-0.0003	0.0001	0.0555
18	8	4.9575	-0.1947	0	0	61.0700	0	0	0	17.2374	0	-0	0
19	6	4.7336	-1.4476	5.8820	0	61.9414	0	0.0043	-0.0086	16.8438	0.0001	0.0001	0.0768
20	6	2.5646	0.3319	0.7261	0	62.7899	0	0.0053	0.0082	18.3022	-0.0083	0.0027	0.0558
...	...	...	...	...	...	...	...	...	...	...	...	...	...
495	8	3.9654	0.0591	-0	0	108.2457	0.0038	0.0049	-0.0023	33.1313	0.0027	0.0009	0.0363
496	5	4.0421	0.0616	0.0001	0.0017	107.5186	0	0	0	33.4224	-0.0005	0.0003	0.0214
497	6	4.0514	0.0127	-0.0002	0.0020	109.8836	0	0	-0.0001	33.3388	0.0002	0	0.0443
498	7	4.1646	0.0442	-0.0012	-0.0053	107.8013	-0.0021	0.0033	0.0038	33.2862	-0.0002	0.0006	0.0343
499	6	4.1226	0.0352	-0.0020	0	108.1554	-0.0005	0.0032	0.0039	33.2862	0.0010	0.0001	0.0068
500	6	4.2625	0.0733	-0.0002	0	110.5101	-0.0032	0.0033	0.0016	36.1647	0.0003	0.0003	0.0079
501	8	3.1551	0	0	-0.0009	110.3071	0.0014	0.0025	0	34.7467	0	0	0
502	4	4.1564	0.0914	-0.0002	0	111.1186	0	0.0013	-0.0031	49.4050	0.0026	-0.0002	0.0211
503	5	4.5799	0.0467	0.0004	0	112.0057	0	0.0027	-0.0043	34.7207	-0.0001	-0.0001	0.0216
504	4	4.3061	0.0236	0.0002	0.0007	112.3186	0	0.0021	0.0015	34.4483	0.0019	0.0003	0.0170
505	9	4.4325	-0.0015	-0.0018	0.0030	106.3345	0	0.0043	0.0001	33.7160	0	-0.0006	0.0265
...	...	...	...	...	...	...	...	...	...	...	...	...	...
1102	7	3.5429	-0.0286	-0.0006	-0.0028	110.3777	0.0006	0.0045	0	5.2399	0	0.0013	0.0008
1103	4	3.5601	0.1028	0.0001	0.0001	111.1585	-0.0003	0.0083	0	5.4824	0	0.0077	0.0485
1104	4	3.5314	0.0809	0.0018	0.0090	109.0996	-0.0007	0.0095	0.0013	11.0949	-0.0018	0.0005	0.1175
1105	4	3.4439	0.1551	-0.0008	-0.0015	106.5667	0.0033	0.0073	-0.0020	4.8300	-0.0012	-0.0003	0.1283
1106	6	3.8206	0.2258	0.0004	0	104.7497	0	0	0.0027	4.8300	0	0.0008	0
1107	4	3.6917	0.2132	-0.0001	-0.0008	105.1229	0.0011	0.0039	0	4.3586	-0.0005	0.0004	0.1418
1108	5	3.7871	0	0	0	105.3595	0.0006	0.0027	-0.0009	4.8000	0.0006	-0.0001	0.1265
1109	4	3.8445	-0.0405	-0.0011	0.0011	102.9022	-0.0044	0.0037	0.0039	5.0279	0	0	0
1110	5	3.8399	0.0212	0.0004	0	102.8313	-0.0020	0.0045	0.0018	4.6817	0.0021	0.0041	0.0536

Table 15 shows the estimates of the ϵ- sub-optimal size $\hat{m}_k$, $j \in I(1,3)$, the parameters $u_1(\hat{m}_k,k)$, $\kappa_{1,1}(\hat{m}_k,k)$, $\kappa_{1,2}(\hat{m}_k,k)$, $\kappa_{1,3}(\hat{m}_k,k)$, $u_2(\hat{m}_k,k)$, $\kappa_{2,1}(\hat{m}_k,k)$, $\kappa_{2,2}(\hat{m}_k,k)$, $\kappa_{2,3}(\hat{m}_k,k)$, $u_3(\hat{m}_k,k)$, $\kappa_{3,1}(\hat{m}_k,k)$, $\kappa_{3,2}(\hat{m}_k,k)$, $\kappa_{3,3}(\hat{m}_k,k)$ for each of the energy commodity data sets. Moreover, $d \leq r$ and the initial real data time is $t_r = t_{10}$.

The following table gives the drift coefficient's parameter estimates $u_1(\hat{m}_k,k)$, $u_2(\hat{m}_k,k)$ and $u_3(\hat{m}_k,k)$ for the decoupled dynamical system for **y** in the case where jump is not incorporated into the dynamical system.

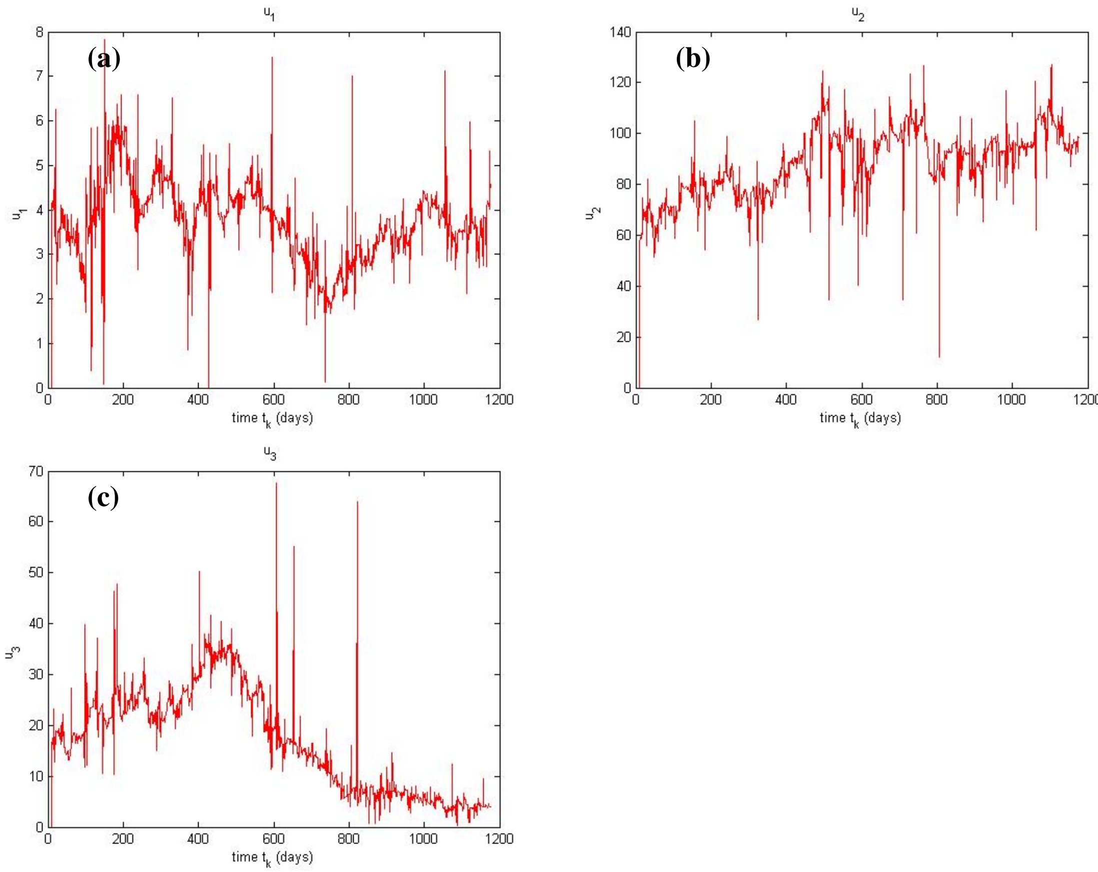
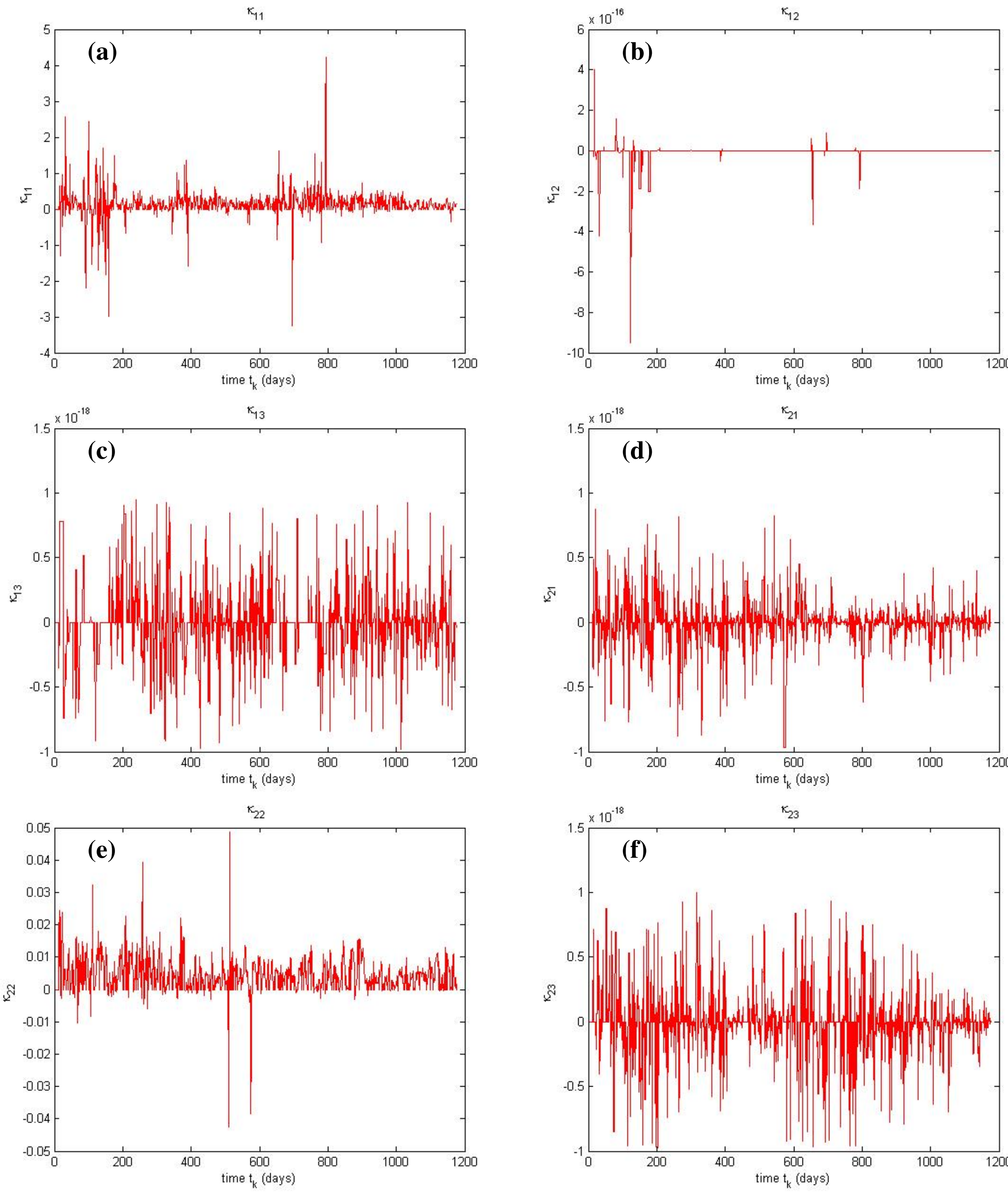


Figure 25.: The graph of mean level $u_1(\hat{m}_k,k)$, $u_2(\hat{m}_k,k)$ and $u_3(\hat{m}_k,k)$ for Natural gas, Crude oil and Coal, respectively (without jump).

Figures 25: **(a), (b) and (c)** are the graphs of $u_1(\hat{m}_k,k)$, $u_2(\hat{m}_k,k)$, and $u_3(\hat{m}_k,k)$ against time t_k for the daily Henry Hub natural gas price [27], daily crude oil price [28], and daily coal price [26] data set, respectively. By plotting the real data sets (shown in Figure 31), it is easily seen that the graphs of $u_1(\hat{m}_k,k)$, $u_2(\hat{m}_k,k)$ and $u_3(\hat{m}_k,k)$ are similar to the graph of the real Henry Hub Natural gas, Crude Oil, and Coal data set, respectively. We expect this to happen because u_j, $j \in I(1,3)$ are the expected equilibrium spot price processes described in (9.3). This analysis shows that the parameters u_j, $j \in I(1,3)$ are statistic process for the respective mean of the data sets at time t_k.

The graph of the parameters $\kappa_{1,1}(\hat{m}_k, k)$, $\kappa_{1,2}(\hat{m}_k, k)$, $\kappa_{1,3}(\hat{m}_k, k)$, $\kappa_{2,1}(\hat{m}_k, k)$, $\kappa_{2,2}(\hat{m}_k, k)$, $\kappa_{2,3}(\hat{m}_k, k)$, $\kappa_{3,1}(\hat{m}_k, k)$, $\kappa_{3,2}(\hat{m}_k, k)$, and $\kappa_{3,3}(\hat{m}_k, k)$ for the decoupled dynamical system for **y** (with no jump incorporated into the dynamical system) are given below:



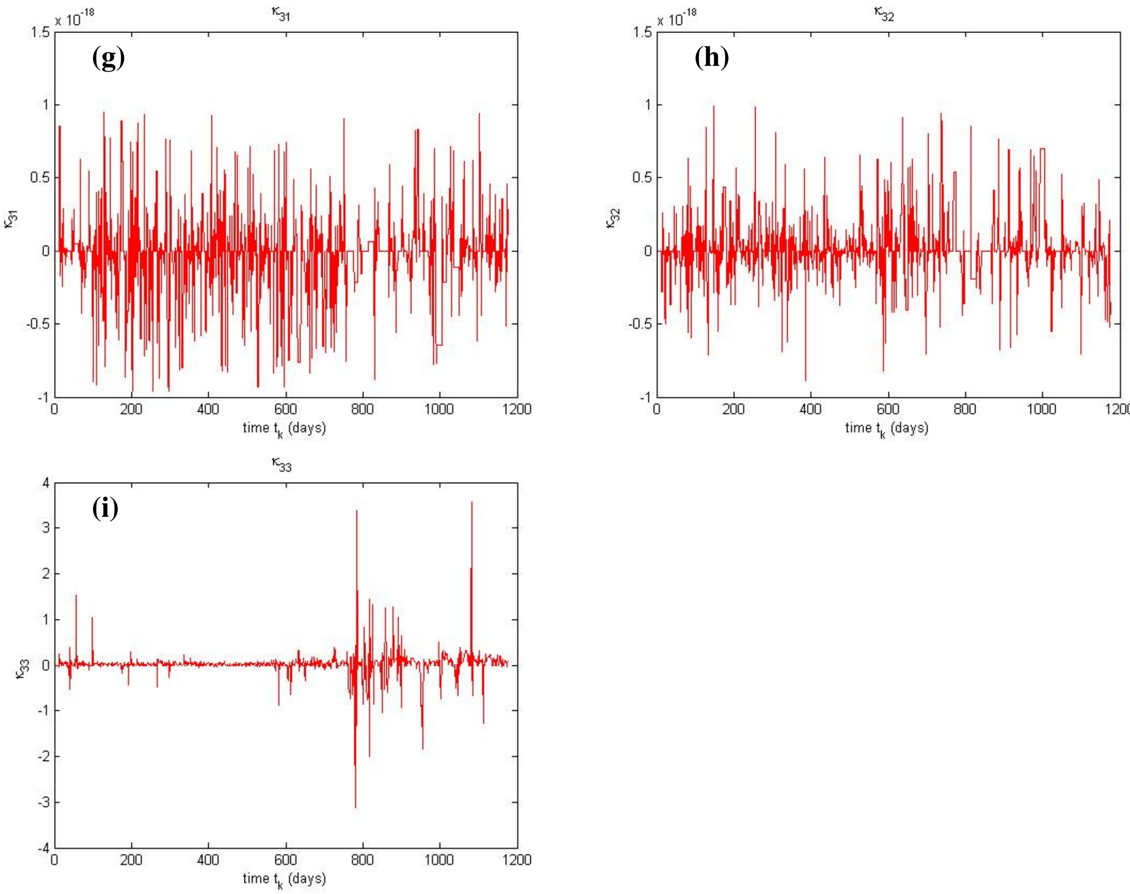


Figure 26.: The graph of interaction coefficients $\kappa_{1,1}(\hat{m}_k, k)$, $\kappa_{1,2}(\hat{m}_k, k)$, $\kappa_{1,3}(\hat{m}_k, k)$, $\kappa_{2,1}(\hat{m}_k, k)$, $\kappa_{2,2}(\hat{m}_k, k)$, $\kappa_{2,3}(\hat{m}_k, k)$, $\kappa_{3,1}(\hat{m}_k, k)$, $\kappa_{3,2}(\hat{m}_k, k)$, $\kappa_{3,3}(\hat{m}_k, k)$ (without jump).

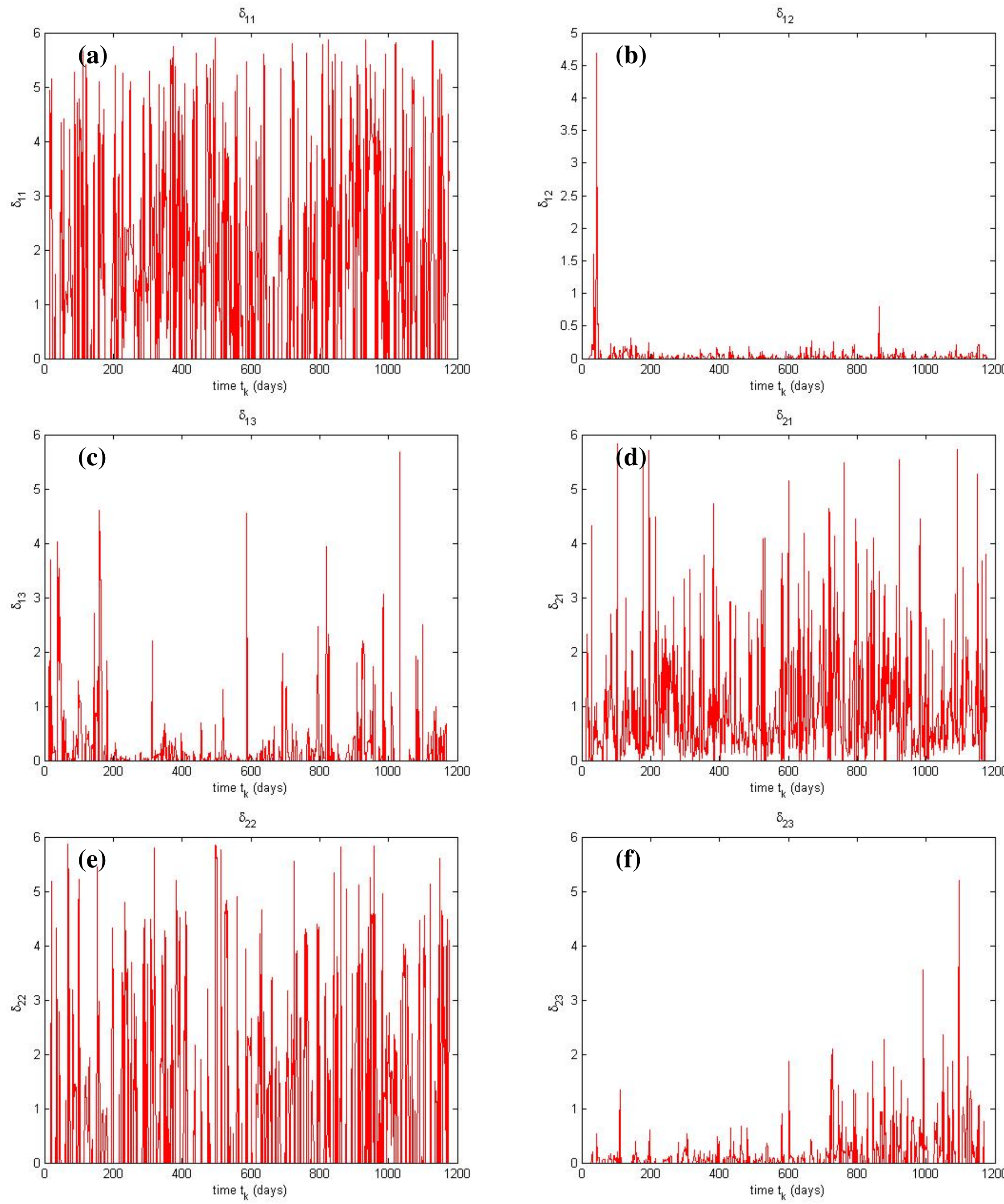
Figures 26 $(a)-(i)$ show the graph of the ϵ- sub-optimal interaction coefficient parameters $\kappa_{1,1}(\hat{m}_k, k)$, $\kappa_{1,2}(\hat{m}_k, k)$, $\kappa_{1,3}(\hat{m}_k, k)$, $\kappa_{2,1}(\hat{m}_k, k)$, $\kappa_{2,2}(\hat{m}_k, k)$, $\kappa_{2,3}(\hat{m}_k, k)$, $\kappa_{3,1}(\hat{m}_k, k)$, $\kappa_{3,2}(\hat{m}_k, k)$, $\kappa_{3,3}(\hat{m}_k, k)$. The interaction coefficients $\kappa_{j,l}$, $j \neq l$ are negligible, because each estimate is $<< 10^{-15}$. Thus, this shows that the model describing the mean spot price, y_j, is mainly characterized by the market potential $\kappa_{j,j}\,(u_j - y_j)\,y_j$, $j \in I(1, n)$.

The table below shows the estimates of the diffusion coefficient's parameters for the model governing $\mathbf{y}$.

Table 16: Estimates $\delta_{1,1}(\hat{m}_k,k)$, $\delta_{1,2}(\hat{m}_k,k)$, $\delta_{1,3}(\hat{m}_k,k)$, $\delta_{2,1}(\hat{m}_k,k)$,$\delta_{2,2}(\hat{m}_k,k)$, $\delta_{2,3}(\hat{m}_k,k)$, $\delta_{3,1}(\hat{m}_k,k)$, $\delta_{3,2}(\hat{m}_k,k)$, $\delta_{3,3}(\hat{m}_k,k)$ (without jump).

t_k	Natural gas			Crude oil			Coal		
	$\delta_{1,1}$	$\delta_{1,2}$	$\delta_{1,3}$	$\delta_{2,1}$	$\delta_{2,2}$	$\delta_{2,3}$	$\delta_{3,1}$	$\delta_{3,2}$	$\delta_{3,3}$
11	0.0123	0.0012	0.0001	0	0.0223	0	0.0412	0	0.0022
12	0.0024	0.0011	0.0121	0.0234	0.0245	0	0	0	0.0112
13	0.0001	1.3425	1.7280	1.9811	0.9899	0.9731	0.6374	0.6374	0.0123
14	0	1.1267	0.6027	2.3258	0.1213	3.9128	1.6564	1.6564	0.0004
15	1.15260	0.4287	0.6210	2.3252	0.0006	0.5083	1.6650	1.6650	0.4565
16	4.9354	0	0	2.3217	0.0120	1.1124	1.6724	1.6724	0.8762
17	4.1360	0.0989	3.6877	1.6425	0	0	1.7719	1.7719	0
18	3.0410	0.1527	0	1.3105	0.9167	1.3451	1.7630	1.7630	0
19	2.7713	0	0	1.1052	0	3.3241	1.7400	1.7400	0
20	2.8461	0.0012	0.2221	0.1196	5.1929	0	0.6532	0.9876	0.0082
....	....	....	....	....	....	....	....	....	....
....	....	....	....	....	....	....	....	....	....
494	2.9961	0.0586	0	0.5529	0	0.42339	0	0	0.5187
495	5.9059	0	0.0584	0.5488	0.8947	0	0.0017	0.0021	0.0001
496	0.1121	0	0.6613	0.5767	0.9899	0	0.8763	0	0.9827
497	1.1229	0.0095	0.0988	0.6499	5.8547	0	1.1317	1.1317	0.0012
498	0.6946	0.0101	0	0	5.8298	0.0320	1.0294	1.0294	0.0321
499	0.7353	0.0066	0.0384	0	5.7180	0.0330	0.7317	0.7317	0.0431
500	1.7509	0.0069	0.0283	0.4307	5.6133	0.0413	0.4826	0.4826	0.0783
501	2.1299	0.0077	0.0282	0.5043	5.6282	0.0308	0.4272	0.4272	0.0002
502	0.9778	0.0077	0.0255	0.2878	4.6543	0.0322	0.5239	0.5239	0.0098
503	0.9872	0	0	0.2909	4.5544	0.0411	1.4523	1.4523	0.0087
504	1.1329	0	0	0.3707	0	0.1128	2.4181	2.4181	0
505	1.9178	0	0	0.3812	1.3243	0.1724	4.9207	4.9207	0
....	....	....	....	....	....	....	....	....	....
....	....	....	....	....	....	....	....	....	....
1102	0	0.0331	0.0056	0.9297	3.9502	0	0.2853	1.8033	1.1355
1103	1.5077	0.0626	0.0332	1.1017	2.8221	0	0	0	1.4133
1104	0	0.0435	0.5821	0.1939	4.5585	0	0	0	1.1672
1105	0	0	1.52970	0.1922	3.2418	0.7273	0.2726	0.2726	0
1106	4.4476	0.323	0.5112	3.5487	3.8113	1.0179	0.3296	0.3296	0
1107	2.4312	0.0011	0.0435	0.2001	2.6026	0.9354	0	0	1.7245
1108	2.5079	0.1232	0.4542	0.3781	0	0.8825	0.1878	0.1878	0
1109	1.7828	0.0431	0.3210	0.4024	0	0.8812	0	0	1.3191
1110	1.2706	0.0056	1.1123	0.3252	0	0.8078	0	0	1.0233

The graph of the diffusion coefficient's parameter for the decoupled dynamical system for **y** without jump incorporated into the dynamical system are given below:



δ11
(a)
δ11
time tk (days)
δ12
(b)
δ12
time tk (days)
δ13
(c)
δ13
time tk (days)
δ21
(d)
δ21
time tk (days)
δ22
(e)
δ22
time tk (days)
δ23
(f)
δ23
time tk (days)

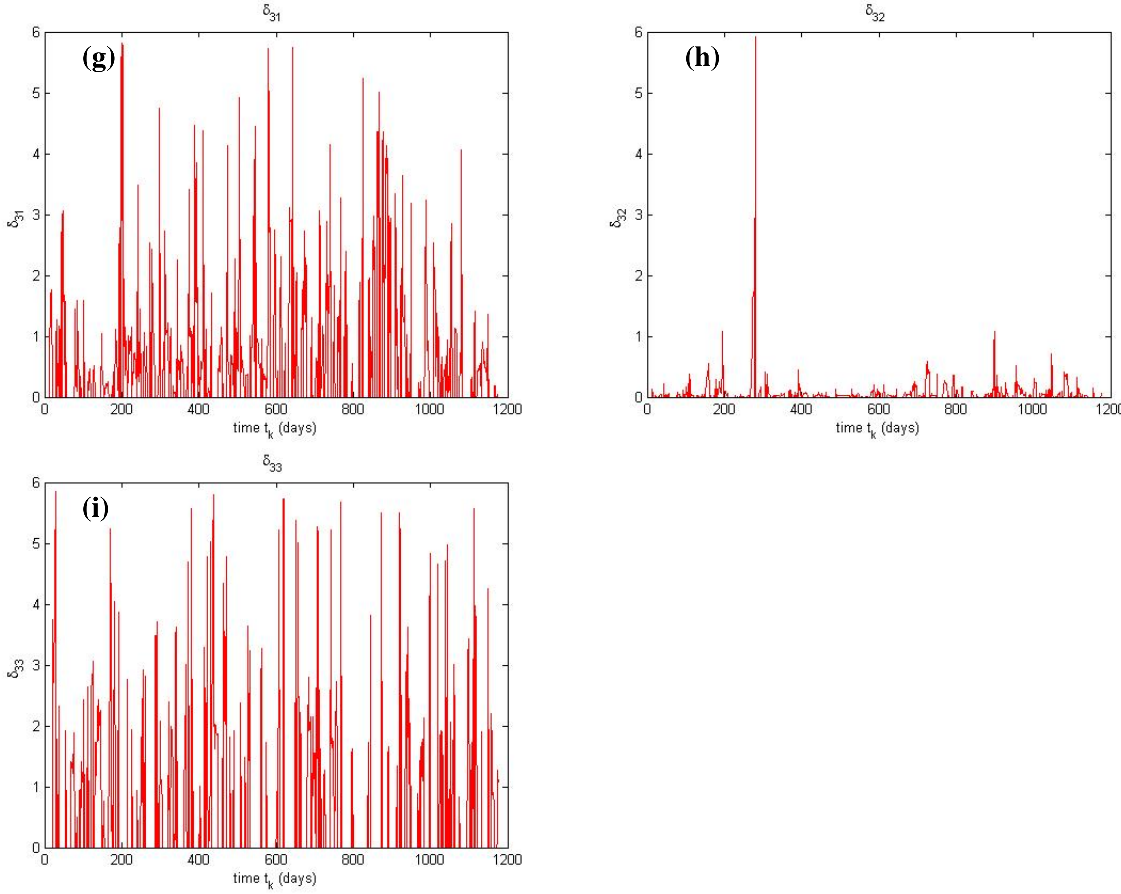


Figure 27.: The graph of interaction coefficients $\delta_{1,1}(\hat{m}_k, k)$, $\delta_{1,2}(\hat{m}_k, k)$, $\delta_{1,3}(\hat{m}_k, k)$, $\delta_{2,1}(\hat{m}_k, k)$, $\delta_{2,2}(\hat{m}_k, k)$, $\delta_{2,3}(\hat{m}_k, k)$, $\delta_{3,1}(\hat{m}_k, k)$, $\delta_{3,2}(\hat{m}_k, k)$, $\delta_{3,3}(\hat{m}_k, k)$ (without jump).

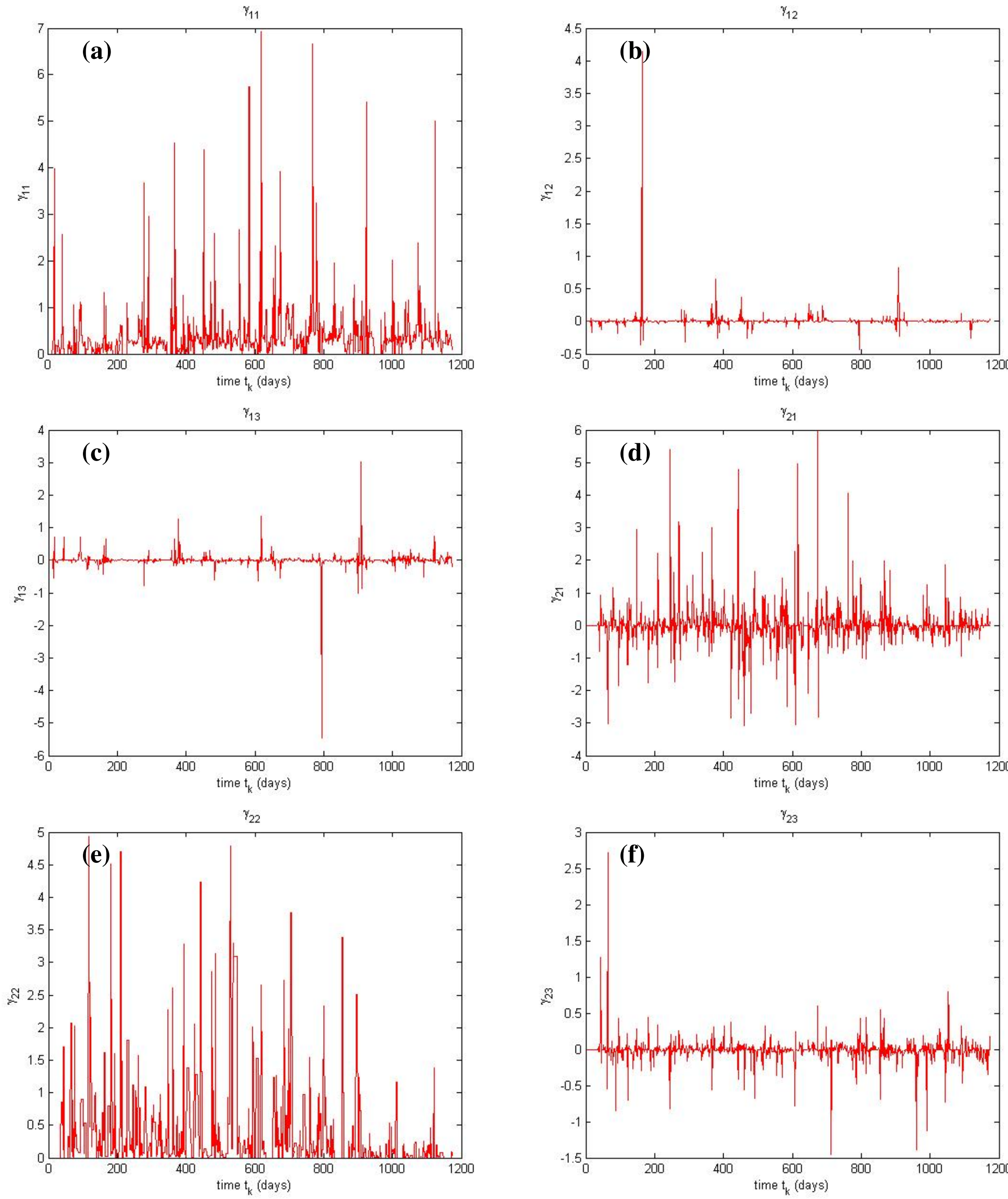
Figures 27 $(a) - (i)$ show the graph of the ϵ- sub-optimal interaction measure of fluctuation coefficient parameters $\delta_{1,1}(\hat{m}_k, k)$, $\delta_{1,2}(\hat{m}_k, k)$, $\delta_{1,3}(\hat{m}_k, k)$, $\delta_{2,1}(\hat{m}_k, k)$, $\delta_{2,2}(\hat{m}_k, k)$, $\delta_{2,3}(\hat{m}_k, k)$, $\delta_{3,1}(\hat{m}_k, k)$, $\delta_{3,2}(\hat{m}_k, k)$, $\delta_{3,3}(\hat{m}_k, k)$, respectively.

The following table gives the drift coefficient's parameter estimates $\beta_1(\hat{m}_k, k)$, $\gamma_{1,1}(\hat{m}_k, k)$, $\gamma_{1,2}(\hat{m}_k, k)$, $\gamma_{1,3}(\hat{m}_k, k)$, $\beta_2(\hat{m}_k, k)$, $\gamma_{2,1}(\hat{m}_k, k)$, $\gamma_{2,2}(\hat{m}_k, k)$, $\gamma_{2,3}(\hat{m}_k, k)$, $\beta_3(\hat{m}_k, k)$, $\gamma_{3,1}(\hat{m}_k, k)$, $\gamma_{3,2}(\hat{m}_k, k)$, and $\gamma_{3,3}(\hat{m}_k, k)$ for the dynamical system for **p** (without incorporating jump process in the model describing the system **p**).

Table 17: Estimates $\beta_1(\hat{m}_k,k)$, $\gamma_{1,1}(\hat{m}_k,k)$, $\gamma_{1,2}(\hat{m}_k,k)$, $\gamma_{1,3}(\hat{m}_k,k)$, $\beta_2(\hat{m}_k,k)$, $\gamma_{2,1}(\hat{m}_k,k)$, $\gamma_{2,2}(\hat{m}_k,k)$, $\gamma_{2,3}(\hat{m}_k,k)$, $\beta_3(\hat{m}_k,k)$, $\gamma_{3,1}(\hat{m}_k,k)$, $\gamma_{3,2}(\hat{m}_k,k)$, $\gamma_{3,3}(\hat{m}_k,k)$ (without jump).

t_k	Natural gas				Crude oil				Coal			
	β_1	$\gamma_{1,1}$	$\gamma_{1,2}$	$\gamma_{1,3}$	β_2	$\gamma_{2,1}$	$\gamma_{2,2}$	$\gamma_{2,3}$	β_3	$\gamma_{3,1}$	$\gamma_{3,2}$	$\gamma_{3,3}$
11	0.3462	0.2579	-0.0039	0.0218	0.0029	0.0023	0.0056	0.0124	0	0	0.0432	0.0012
12	0.1681	0.3497	-0.0109	0.0248	0	0	0	0	0.6812	-0.3513	0.0248	0
13	0.1592	0.3755	-0.0102	0.0228	-0.0490	0.0228	0	-0.0027	1.0795	-0.2904	0.0135	0
14	-0.0889	0.5001	-0.0069	0.0257	-0.3700	-0.1689	0	0.0633	-0.6689	-0.1969	0.0166	0.0177
15	0.7025	0.6513	-0.0242	0.0376	-0.0903	-0.1737	0	0.0477	-0.8220	-0.0944	0.0061	0.0291
16	0.6727	0.6513	0.0188	-0.1048	0.3425	-0.1847	0	0.0252	-0.0480	0.0694	-0.0112	0.0165
17	0.3253	0.3674	-0.0103	0.0140	-0.4531	-0.1457	0	0.0638	-3.6240	0.1086	0.0266	0.0569
18	0.1523	0.3433	0.0014	-0.0163	4.0859	0.2320	0.3114	-0.2889	-1.3277	-0.0566	0.0088	0.0353
19	-7.9573	0.3433	-0.2058	1.1496	0.1389	-0.0677	0.0004	0.0065	-0.9232	0.0368	-0.0277	0.0883
20	0.0514	0.3028	0.0041	-0.0192	-0.1464	-0.0409	0.0004	0.0178	-1.9138	0.0530	-0.0104	0.0836
...	....	....	....	....	....	.....	....	....	....	....	....	....
495	0.0552	0.2994	0.0015	-0.0063	1.8435	-0.1608	0.1040	-0.0368	0.5815	-0.3597	-0.0050	0.0350
496	0.1799	0.1860	0.0004	-0.0067	1.5682	-0.0268	0.4723	-0.0502	0.8586	-0.2029	-0.0094	0.0241
497	0.8047	0.1923	0.0023	-0.0314	6.6699	-0.2655	0.4723	-0.1649	0.7191	-0.2061	-0.0053	0.0172
498	0.2742	0.2651	0.0020	-0.0145	1.0042	0.0088	0.0226	-0.0315	0.3978	-0.1680	-0.0034	0.0161
499	0.4915	0.2295	-0.0006	-0.0125	1.3074	0.3761	0.0073	-0.0872	0.1425	-0.1899	-0.0026	0.0224
500	0.5659	0.1618	0.0008	-0.0194	0.4040	-0.0889	0.0392	-0.0011	0.3674	-0.2313	-0.0073	0.0331
501	0.4498	0.1679	0.0010	-0.0167	0.4230	-0.1297	0.0434	0.0035	-0.9002	1.1703	-0.0775	0.1011
502	0.4836	0.1850	-0.0001	-0.0139	0.5570	-0.1502	0.0384	0.0022	-0.2313	0.6524	-0.0496	0.0663
503	0.4696	0.1850	0.1224	-0.0919	-0.0441	0.0299	0.0384	-0.0023	3.7804	0.0120	-0.0498	0.0389
504	-0.0456	0.1850	0.0088	-0.0270	0.6112	-0.0820	0.0425	-0.0080	6.4696	0.4005	-0.0950	0.0543
505	0.0464	1.7125	-0.0423	0.1339	0.7135	-0.1115	0.1135	-0.0082	2.2295	0.0897	-0.0357	0.0306
...	....	....	....	....	....	....	....	....	....	....	....	....
1102	0.6765	0.0455	-0.0020	-0.0908	0.2863	-0.2183	0.1891	0.1028	4.3927	3.8144	0.1072	0.0250
1103	1.1804	0.4214	-0.0149	0.0837	-2.1858	0.4491	0.1891	0.1135	-6.1960	0.7446	0.0261	0.0144
1104	0.1069	0.2489	-0.0009	-0.0014	-2.1178	0.3406	0.1959	0.1826	-6.4415	0.0339	-0.0037	0.1429
1105	0.2367	0.0128	0.0065	-0.0019	0.2633	0.0565	0.0742	-0.0997	0.6510	2.4930	0.0714	0.1429
1106	0.1178	0	0	-0.0014	0.1384	0.0784	0.2014	-0.0904	0.6510	2.4930	0.6341	0.1429
1107	0.1466	0.4648	-0.0002	-0.0271	0.5787	-0.0297	0.1305	-0.0979	-4.5897	-0.0971	0.0240	0.0509
1108	0.3240	0.2478	0.1212	-0.0074	0.4293	-0.0678	0.0721	-0.0389	-4.5961	-0.0734	0.0233	0.0507
1109	0.121	0	0	-0.0021	0.2282	-0.0468	0.0721	-0.0133	-4.5961	-2.6776	0.5625	0.4702
1110	0.002	0	0	-0.0056	0.1523	0.0121	0.0011	-0.0129	9.5959	0.9045	-0.1478	0.0499

Table 17 shows the estimates of the parameters $\beta_1(\hat{m}_k,k)$, $\gamma_{1,1}(\hat{m}_k,k)$, $\gamma_{1,2}(\hat{m}_k,k)$, $\gamma_{1,3}(\hat{m}_k,k)$, $\beta_2(\hat{m}_k,k)$, $\gamma_{2,1}(\hat{m}_k,k)$, $\gamma_{2,2}(\hat{m}_k,k)$, $\gamma_{2,3}(\hat{m}_k,k)$, $\beta_3(\hat{m}_k,k)$, $\gamma_{3,1}(\hat{m}_k,k)$, $\gamma_{3,2}(\hat{m}_k,k)$, $\gamma_{3,3}(\hat{m}_k,k)$ at the ϵ- sub-optimal size $\hat{m}_k$ and time t_k, for each of the energy commodity data sets. Moreover, $p \leq r$, and the initial real data time is $t_r = t_{10}$.



γ11
(a)
γ11
time t_k (days)
γ12
(b)
γ12
time t_k (days)
γ13
(c)
γ13
time t_k (days)
γ21
(d)
γ21
time t_k (days)
γ22
(e)
γ22
time t_k (days)
γ23
(f)
γ23
time t_k (days)

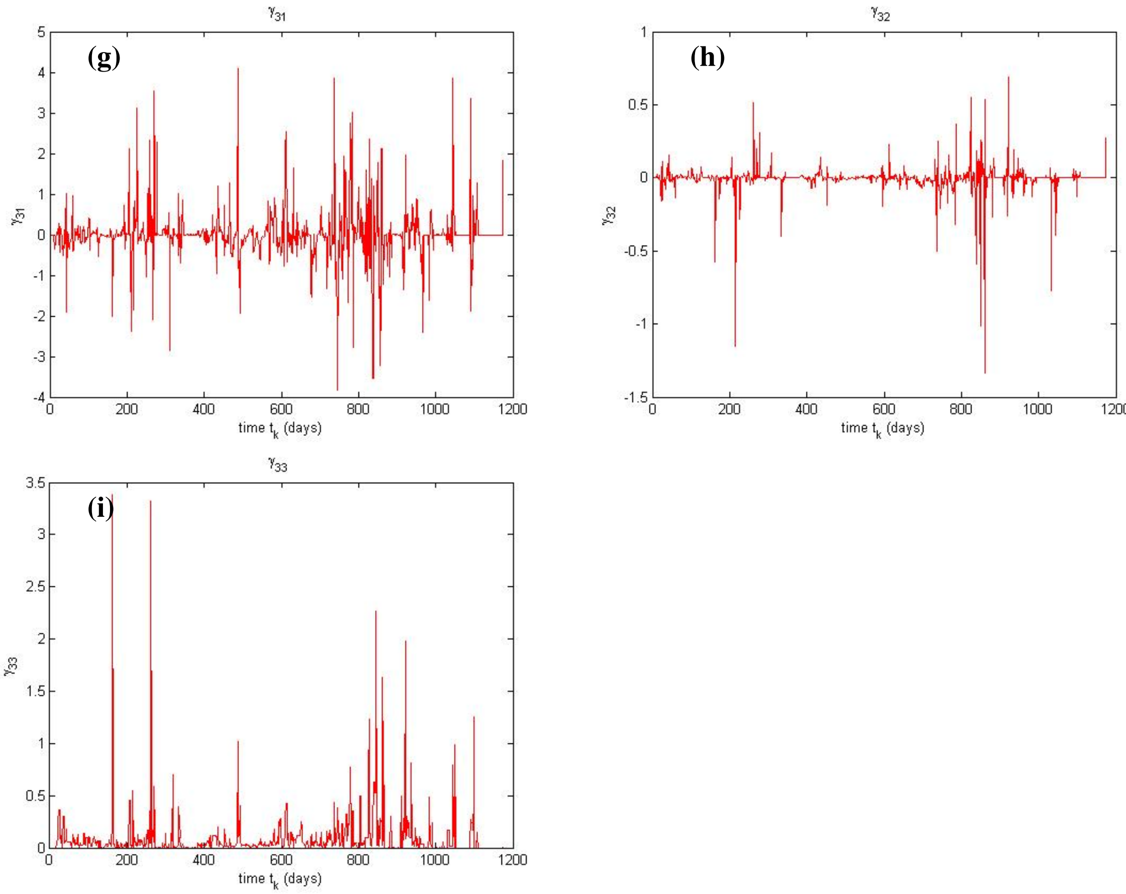


Figure 28.: The graph of interaction coefficients $\gamma_{1,1}(\hat{m}_k, k)$, $\gamma_{1,2}(\hat{m}_k, k)$, $\gamma_{1,3}(\hat{m}_k, k)$, $\gamma_{2,1}(\hat{m}_k, k)$, $\gamma_{2,2}(\hat{m}_k, k)$, $\gamma_{2,3}(\hat{m}_k, k)$, $\gamma_{3,1}(\hat{m}_k, k)$, $\gamma_{3,2}(\hat{m}_k, k)$, $\gamma_{3,3}(\hat{m}_k, k)$ (without jump).

Figures 28 $(a) - (i)$ show the graph of the ϵ- sub-optimal interaction coefficient parameters $\gamma_{1,1}(\hat{m}_k, k)$, $\gamma_{1,2}(\hat{m}_k, k)$, $\gamma_{1,3}(\hat{m}_k, k)$, $\gamma_{2,1}(\hat{m}_k, k)$, $\gamma_{2,2}(\hat{m}_k, k)$, $\gamma_{2,3}(\hat{m}_k, k)$, $\gamma_{3,1}(\hat{m}_k, k)$, $\gamma_{3,2}(\hat{m}_k, k)$, $\gamma_{3,3}(\hat{m}_k, k)$ without jump. According to (9.47), the estimate $\gamma_{j,l}(\hat{m}_k, k)$, $j \neq l$, is positive if commodity p_l is cooperating with commodity p_j, and negative if commodity p_l is competing with commodity p_j. There is no interaction between the two commodities if $\gamma_{j,l}(\hat{m}_k, k) = 0$. It is apparent from the graph of $\gamma_{1,3}(\hat{m}_k, k)$ that coal and natural gas are competing and cooperating depending on the time period. It is also apparent graph of $\gamma_{1,2}(\hat{m}_k, k)$ that natural gas and crude oil are also either cooperating or competing, depending on the time period.

The next figure shows the graph of the parameter estimates $\beta_1(\hat{m}_k, k)$, $\beta_2(\hat{m}_k, k)$ and $\beta_3(\hat{m}_k, k)$ in the drift coefficient of the model describing the system **p**.

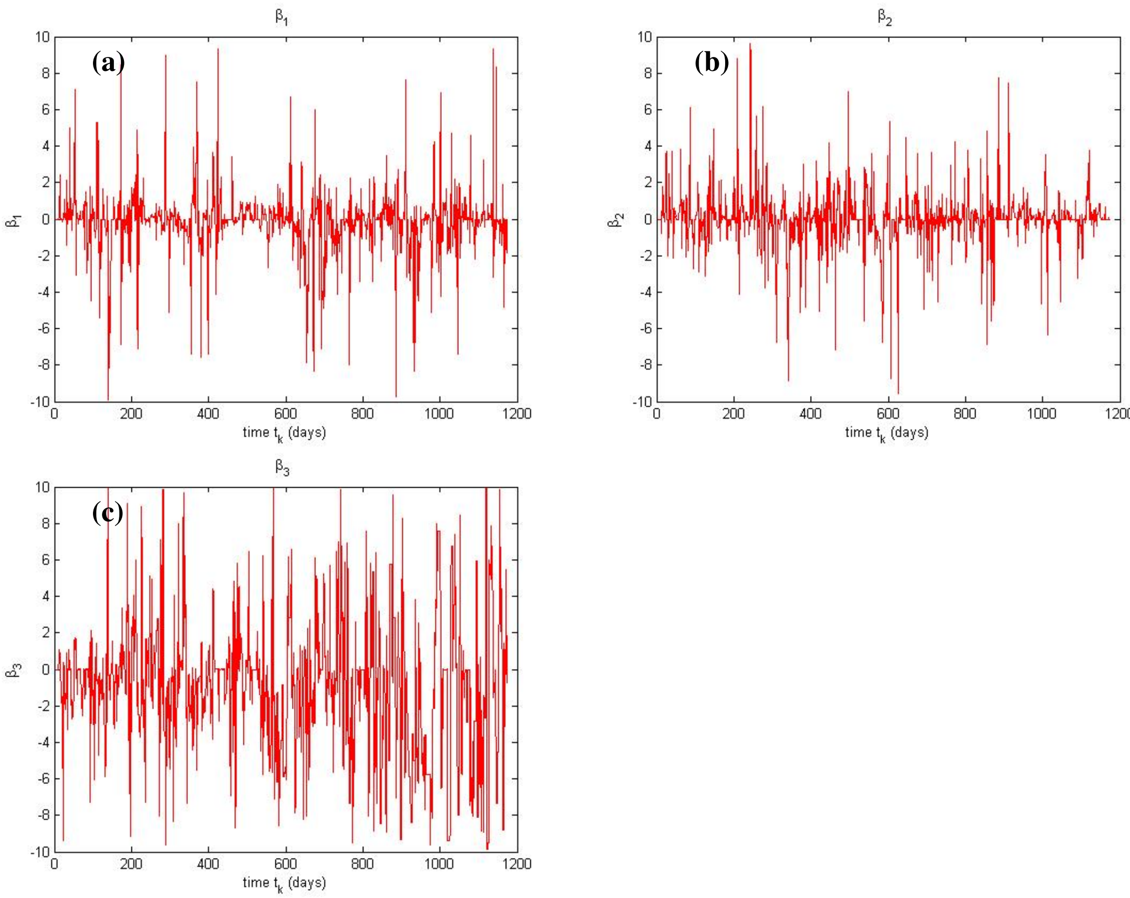


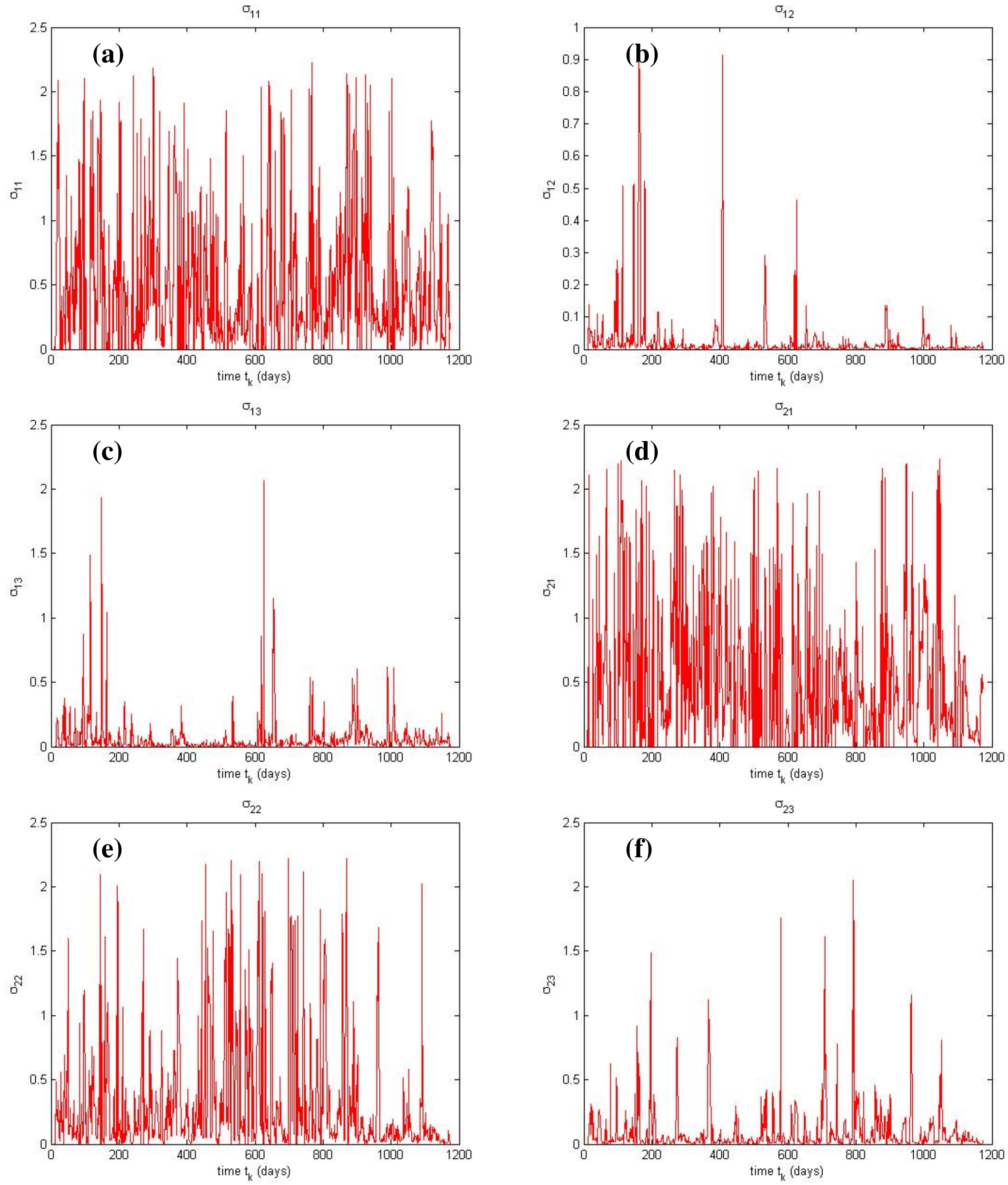
Figure 29.: The graph of $\beta_1(\hat{m}_k, k)$, $\beta_2(\hat{m}_k, k)$ and $\beta_3(\hat{m}_k, k)$ for Natural gas, Crude oil and Coal, respectively (without jump).

Figures 29: **(a), (b) and (c)** are the graphs of $\beta_1(\hat{m}_k, k)$, $\beta_2(\hat{m}_k, k)$ and $\beta_3(\hat{m}_k, k)$ against time t_k for the daily Henry Hub natural gas price data set [27], daily crude oil price data set [28], and daily coal price data set, respectively (without jump).

Table 18: Estimates $\sigma_{1,1}(\hat{m}_k, k)$, $\sigma_{1,2}(\hat{m}_k, k)$, $\sigma_{1,3}(\hat{m}_k, k)$, $\sigma_{2,1}(\hat{m}_k, k)$, $\sigma_{2,2}(\hat{m}_k, k)$, $\sigma_{2,3}(\hat{m}_k, k)$, $\sigma_{3,1}(\hat{m}_k, k)$, $\sigma_{3,2}(\hat{m}_k, k)$, $\sigma_{3,3}(\hat{m}_k, k)$ (without jump).

t_k	Natural gas			Crude oil			Coal		
	$\sigma_{1,1}$	$\sigma_{1,2}$	$\sigma_{1,3}$	$\sigma_{2,1}$	$\sigma_{2,2}$	$\sigma_{2,3}$	$\sigma_{3,1}$	$\sigma_{3,2}$	$\sigma_{3,3}$
11	0	0	0	0	0	0	0	0	0
12	0.0485	0.0004	0.0032	0.2734	0.0166	0	0	0	0.0000
13	0	0	0	0	0	0	0	0	0
14	0.2120	0.1386	0.0133	1.2573	0.4773	0.1195	0	0.0665	0.0086
15	0.4246	0.1318	0.0021	2.1081	0.4894	0.1211	0	0.6107	0.0696
16	0.5538	0.0778	0.1501	0	0.2524	0.0811	0.0651	0.4251	0.0635
17	1.1121	0.0469	0.2230	0	0.1848	0.2463	0	0.4458	0.0478
18	1.5347	0.0180	0.2178	0	0.1877	0.1602	0.5681	0.0592	0.0115
19	1.1315	0.0619	0.2221	0	0.2673	0.2465	0.4999	0.0569	0.0127
20	2.0845	0.0536	0.1866	0	0.1700	0.0781	0.3789	0.3174	0.0046
...	..	...	...	...	...	...	...	...	...
495	0	0.0036	0.0406	0.2286	0.0600	0.0172	0	0.9387	0.0182
496	0.1588	0.0035	0.0107	1.4847	0.3163	0.0102	0	0	0.0016
497	0.1551	0.0009	0.0065	0	0.1453	0	0.7777	0	0.0033
498	0.1576	0.0011	0.0073	0	0.1679	0	0.5334	0	0.0060
499	0.1197	0.0006	0.0059	1.9414	0.2391	0.0172	0.4405	0.1432	0.0097
500	0.3600	0.0001	0.0049	1.9554	0.3960	0.0079	0.6331	0.1410	0.0093
501	0.0514	0.0033	0.0049	2.0436	0.3499	0.0111	0.7690	0.1376	0.0089
502	0.2503	0.0034	0.0042	2.0837	0.1744	0.0132	0.6198	0.1274	0.0066
503	0.1195	0.0147	0.0165	0	0.4283	0.0060	1.1613	0.1530	0.0049
504	0.0974	0.0144	0.0027	0	0.2241	0.0048	0.4778	0.0574	0.0043
505	0.1422	0.0060	0.0131	0	0.2023	0.0054	0.5604	0.0669	0.0004
...	..	...	...	...	...	...	...	...	...
1102	0.1898	0.0016	0.0413	0.8313	0.0767	0.0381	0.6875	0	0.1451
1103	0.2094	0.0015	0.0352	0.8262	0.0673	0.0451	0.7298	0.2808	0.0147
1104	0.1711	0.0011	0.0040	0.6648	0.0915	0.0462	0.5563	0.1831	0.0105
1105	0.1816	0.0012	0.0116	0.6658	0.1049	0.0371	0.6591	0.2874	0.0057
1106	0.1191	0.0011	0.0116	0.6260	0.1155	0.0393	0	0.0196	0.0060
1107	0.0417	0.0012	0.0041	0.4992	0.0781	0.0382	0	0	0.0065
1108	0.1058	0.0033	0.0045	0.0019	0.0589	0.0421	0	0	0.0018
1109	0.1740	0.0021	0	0	0.0446	0.0316	2.1187	0	0.4511
1110	0.2912	0.0021	0.0163	0.0385	0.0342	0.0037	0	1.1563	0.0257

Table 18 gives the ϵ-sub-optimal estimates of the parameters $\sigma_{1,1}(\hat{m}_k, k)$, $\sigma_{1,2}(\hat{m}_k, k)$, $\sigma_{1,3}(\hat{m}_k, k)$, $\sigma_{2,1}(\hat{m}_k, k)$, $\sigma_{2,2}(\hat{m}_k, k)$, $\sigma_{2,3}(\hat{m}_k, k)$, $\sigma_{3,1}(\hat{m}_k, k)$, $\sigma_{3,2}(\hat{m}_k, k)$, $\sigma_{3,3}(\hat{m}_k, k)$ for each of the energy commodity data sets.



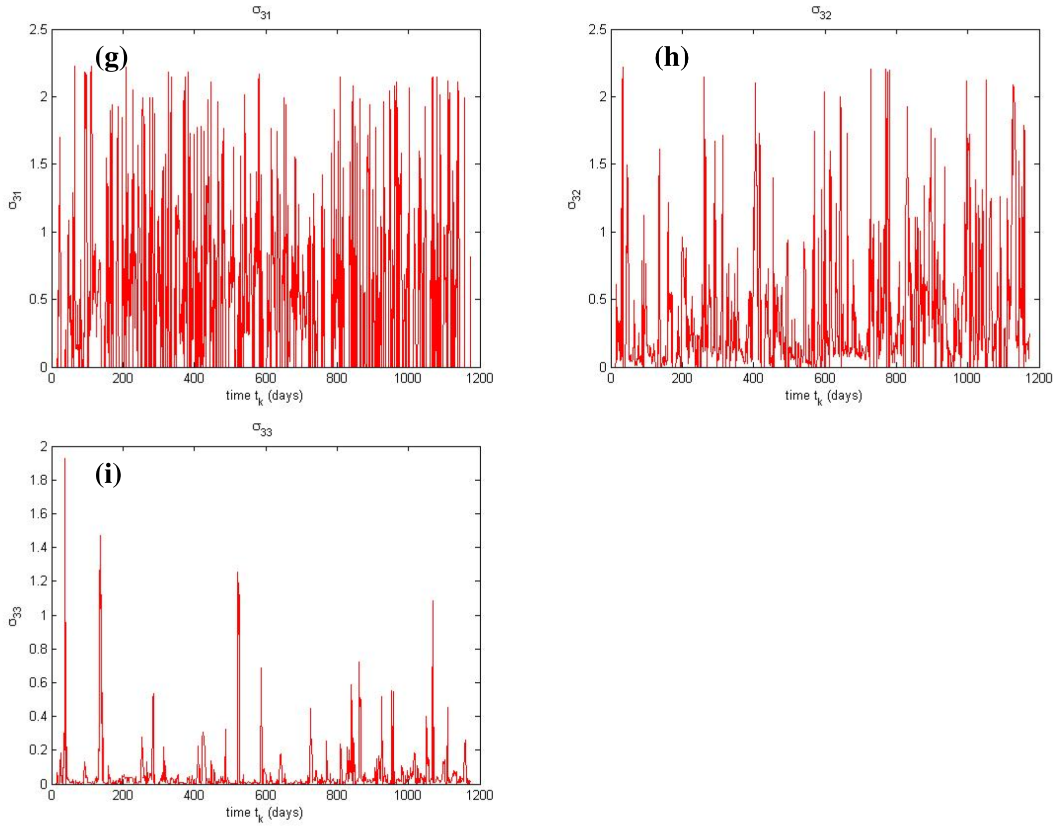


Figure 30.: The graph of $\sigma_{1,1}(\hat{m}_k, k)$, $\sigma_{1,2}(\hat{m}_k, k)$, $\sigma_{1,3}(\hat{m}_k, k)$, $\sigma_{2,1}(\hat{m}_k, k)$, $\sigma_{2,2}(\hat{m}_k, k)$, $\sigma_{2,3}(\hat{m}_k, k)$, $\sigma_{3,1}(\hat{m}_k, k)$, $\sigma_{3,2}(\hat{m}_k, k)$, $\sigma_{1,1}(\hat{m}_k, k)$ for Natural gas, Crude oil and Coal, respectively (without jump).

Figures 30: **(a), (b) and (c)** are the graphs of $\sigma_{1,1}(\hat{m}_k, k)$, $\sigma_{1,2}(\hat{m}_k, k)$, $\sigma_{1,3}(\hat{m}_k, k)$, $\sigma_{2,1}(\hat{m}_k, k)$, $\sigma_{2,2}(\hat{m}_k, k)$, $\sigma_{2,3}(\hat{m}_k, k)$, $\sigma_{3,1}(\hat{m}_k, k)$, $\sigma_{3,2}(\hat{m}_k, k)$, $\sigma_{1,1}(\hat{m}_k, k)$ against time t_k for the daily Henry Hub natural gas price data set [27], daily crude oil price data set [28], and daily coal price data set, respectively.

Table 19: Real and simulated estimates (without jump) for Natural gas, Crude oil, and Coal.

t_k	Natural gas		Crude oil		Coal	
	Real	Simulated p_1^s	Real	Simulated p_2^s	Real	Simulated p_3^s
11	4.0200	4.0500	58.9900	56.5200	16.5900	16.8000
12	3.9900	4.0500	59.5200	59.3099	17.4600	16.8635
13	3.7500	3.6690	61.4500	59.3377	17.8900	17.8086
14	3.7700	3.6341	60.4900	59.4191	17.5500	17.0859
15	3.4100	3.3967	61.1500	59.6974	17.4100	17.0859
16	3.3500	3.3967	62.4800	59.6974	16.7500	17.0859
17	3.4900	3.4537	63.4100	61.2177	17.6600	19.0677
18	3.5500	3.4537	65.0900	61.4561	17.5200	16.0578
19	3.9200	3.8618	66.3100	61.6529	18.5000	19.0677
20	3.8600	3.8618	68.5900	60.9364	19.0600	19.0677
...	...	...	...	...	...	...
495	4.1900	4.0368	107.1800	104.1295	32.7600	31.3108
496	4.3300	4.1868	110.8400	111.1245	33.6500	32.7737
497	4.3300	4.1025	111.7200	112.4675	33.7100	33.4888
498	4.3700	4.0964	111.6800	110.8795	34.7500	35.5907
499	4.3200	4.1042	111.7200	104.2465	34.5400	32.9391
500	4.3500	4.0548	112.3100	109.9535	34.0400	36.2674
501	4.3800	4.0548	112.3800	109.9995	33.1000	36.2674
502	4.5100	4.3249	113.3900	104.3254	33.6700	34.8915
503	4.6000	4.3555	113.0300	113.2356	33.9400	35.0472
504	4.6000	4.3491	110.6000	103.9435	33.8300	32.8992
505	4.5900	4.3609	108.7900	104.9995	32.0200	32.8992
...	...	...	...	...	...	...
1102	3.7200	3.5963	108.2300	110.5149	4.7700	2.8861
1103	3.7300	3.5963	106.2600	105.8076	5.0100	5.6871
1104	3.6800	3.4099	104.7000	105.8076	4.9800	5.3821
1105	3.6600	3.4356	103.6200	105.8076	4.7300	4.9221
1106	3.5900	3.4636	103.2200	106.9547	4.6800	4.2352
1107	3.5200	3.2573	102.6800	105.4047	4.6300	5.8172
1108	3.4900	2.8981	103.1000	102.4928	4.7400	6.0376
1109	3.5100	2.8981	102.8600	102.4928	4.3300	5.1121
1110	3.4800	3.0267	102.3600	102.4928	4.1800	4.8978

Table 19 shows the Real and simulated estimates for the spot price processes $p_j(t)$, $j \in I(1,3)$ corresponding to the natural gas, crude oil and coal prices.

The next figure shows the graph of the real and simulated prices for Natural gas, Crude oil, and Coal data set.

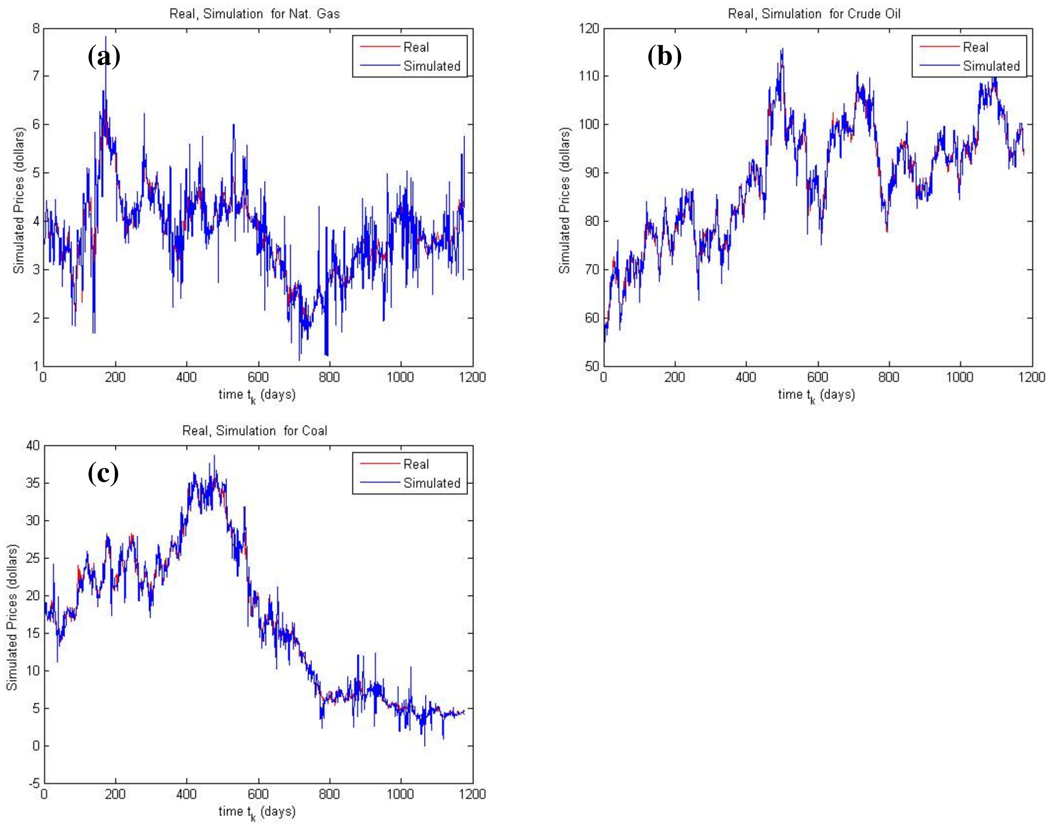


Figure 31.: Real and Simulated Prices (without jump) for Natural gas, Crude oil, and Coal.

Figures 31: **(a), (b), and (c)** show the graph of the Real and Simulated Spot Prices for the daily Henry Hub natural gas data set [27], daily crude oil data set [28], and daily coal data set [26], respectively. The red line represents the real data set $\mathbf{p}(t_k)$, while the blue line represent the simulated data set $\mathbf{p}^s(\hat{m}_k, k)$. Here, we begin by using a starting delay of $r = 10$. The simulation starts from $t_r = t_{10}$. The spikes in the graph is as a result of jump. The estimates at the jump times are not fitted properly. To reduce magnitude of error, we increase the magnitude of time delay. We later compare this result with the case where jump is incorporated into the system.

10.6.2 Relationship between Natural Gas, Crude Oil and Coal: With Jump Incorporated.

In this subsubsection, we analyze the relationship between Natural Gas, Crude Oil, and Coal with the jump process. Here, we apply the above conceptual computational algorithm in Section 10 for the real time data sets namely daily Henry Hub Natural gas data set, daily crude oil data set, and daily coal data set for the period of $05/04/2009 - 01/03/2014$, [26, 27, 28]. For $i \in I(1, K^*)$, $K \neq 0$, we use $\Delta t_{i-1} = 1$; $\epsilon = 0.001$; $r_{i-1} = 10$ and $d_{i-1} = 2$. The $\epsilon-$ level sub-optimal estimates of the parameters at each real data times are described below for each commodity data sets. We also note that there are $K = 15$ jumps in the system.

The parameters corresponding to the model governing natural gas price data set are $u_1^{i-1}(\hat{m}_k, k)$, $\beta_1^{i-1}(\hat{m}_k^{i-1}, t_k^{i-1})$, $\kappa_{1,1}^{i-1}(\hat{m}_k^{i-1}, t_k^{i-1})$, $\kappa_{1,2}^{i-1}(\hat{m}_k^{i-1}, t_k^{i-1})$, $\kappa_{1,3}^{i-1}(\hat{m}_k^{i-1}, t_k^{i-1})$, $\gamma_{1,1}^{i-1}(\hat{m}_k^{i-1}, t_k^{i-1})$, $\gamma_{1,2}^{i-1}(\hat{m}_k^{i-1}, t_k^{i-1})$, $\gamma_{1,3}^{i-1}(\hat{m}_k^{i-1}, t_k^{i-1})$, $\delta_{1,1}^{i-1}(\hat{m}_k^{i-1}, t_k^{i-1})$, $\delta_{1,2}^{i-1}(\hat{m}_k^{i-1}, t_k^{i-1})$, $\delta_{1,3}^{i-1}(\hat{m}_k^{i-1}, t_k^{i-1})$, $\sigma_{1,1}^{i-1}(\hat{m}_k^{i-1}, t_k^{i-1})$, $\sigma_{1,2}^{i-1}(\hat{m}_k^{i-1}, t_k^{i-1})$, $\sigma_{1,3}^{i-1}(\hat{m}_k^{i-1}, t_k^{i-1})$. The parameters corresponding to the model governing crude oil price data set are $u_2^{i-1}(\hat{m}_k^{i-1}, t_k^{i-1})$, $\beta_2^{i-1}(\hat{m}_k^{i-1}, t_k^{i-1})$, $\kappa_{2,1}^{i-1}(\hat{m}_k^{i-1}, t_k^{i-1})$, $\kappa_{2,2}^{i-1}(\hat{m}_k^{i-1}, t_k^{i-1})$, $\kappa_{2,3}^{i-1}(\hat{m}_k^{i-1}, t_k^{i-1})$, $\gamma_{2,1}^{i-1}(\hat{m}_k^{i-1}, t_k^{i-1})$, $\gamma_{2,2}^{i-1}(\hat{m}_k^{i-1}, t_k^{i-1})$, $\gamma_{2,3}^{i-1}(\hat{m}_k^{i-1}, t_k^{i-1})$, $\delta_{2,1}^{i-1}(\hat{m}_k^{i-1}, t_k^{i-1})$, $\delta_{2,2}^{i-1}(\hat{m}_k^{i-1}, t_k^{i-1})$, $\delta_{2,3}^{i-1}(\hat{m}_k^{i-1}, t_k^{i-1})$, $\sigma_{2,1}^{i-1}(\hat{m}_k^{i-1}, t_k^{i-1})$, $\sigma_{2,2}^{i-1}(\hat{m}_k^{i-1}, t_k^{i-1})$, $\sigma_{2,3}^{i-1}(\hat{m}_k^{i-1}, t_k^{i-1})$, while the parameters corresponding to the model governing coal price data set are $u_3^{i-1}(\hat{m}_k^{i-1}, t_k^{i-1})$, $\beta_3^{i-1}(\hat{m}_k^{i-1}, t_k^{i-1})$, $\kappa_{3,1}^{i-1}(\hat{m}_k^{i-1}, t_k^{i-1})$, $\kappa_{3,2}^{i-1}(\hat{m}_k^{i-1}, t_k^{i-1})$, $\kappa_{3,3}^{i-1}(\hat{m}_k^{i-1}, t_k^{i-1})$, $\gamma_{3,1}^{i-1}(\hat{m}_k^{i-1}, t_k^{i-1})$, $\gamma_{3,2}^{i-1}(\hat{m}_k^{i-1}, t_k^{i-1})$, $\gamma_{3,3}^{i-1}(\hat{m}_k^{i-1}, t_k^{i-1})$, $\delta_{3,1}^{i-1}(\hat{m}_k^{i-1}, t_k^{i-1})$, $\delta_{3,2}^{i-1}(\hat{m}_k^{i-1}, t_k^{i-1})$, $\delta_{3,3}^{i-1}(\hat{m}_k^{i-1}, t_k^{i-1})$, $\sigma_{3,1}^{i-1}(\hat{m}_k^{i-1}, t_k^{i-1})$, $\sigma_{3,2}^{i-1}(\hat{m}_k^{i-1}, t_k^{i-1})$, $\sigma_{3,3}^{i-1}(\hat{m}_k^{i-1}, t_k^{i-1})$.

For the sake of simplicity and in order to be able to compare our results in this subsection with the results in subsection 10.6.1, for each $j, l \in I(1, n)$, we re-write the parameters $u_j^{i-1}(\hat{m}_k^{i-1}, t_k^{i-1})$, $\beta_j^{i-1}(\hat{m}_k^{i-1}, t_k^{i-1})$, $\kappa_{j,l}^{i-1}(\hat{m}_k^{i-1}, t_k^{i-1})$, $\gamma_{j,l}^{i-1}(\hat{m}_k^{i-1}, t_k^{i-1})$, $\delta_{j,l}^{i-1}(\hat{m}_k^{i-1}, t_k^{i-1})$, and $\sigma_{j,l}^{i-1}(\hat{m}_k^{i-1}, t_k^{i-1})$ after they have been estimated as $u_j(\hat{m}_k, k)$, $\beta_j(\hat{m}_k, k)$, $\kappa_{j,l}(\hat{m}_k, k)$, $\gamma_{j,l}(\hat{m}_k, k)$, $\delta_{j,l}(\hat{m}_k, k)$, and $\sigma_{j,l}(\hat{m}_k, k)$.

First, we give results for the jump times of the system $\{T_i\}_{i \in I(1,K^*)}$.

Table 20: Result for the jump times of the system $(\mathbf{y}, \mathbf{p})$

T	17	44	61	87	157	200	422	464	483	502	722	754	870	930	1113

Table 20 shows the result for the jump times of the system $(\mathbf{y}, \mathbf{p})$. These results are derived by recording the times at which an entry of a commodity differ by twice the standard deviation or more from the mean of that commodity. These times are now combined into a single array and sorted out in an increasing order. It follows from Table 20 that $K = 15$.

We give the estimate of the jump coefficient matrices Π^i and Θ^i defined in (9.23) in the following table.

Table 21: Estimates π_1^i, π_2^i, π_3^i, θ_1^i, θ_2^i, and θ_3^i.

	T_i	Natural gas	Crude oil	Coal	T_i	Natural gas	Crude oil	Coal
i	T_i	Π_1^i	Π_2^i	Π_3^i	T_i	θ_1^i	θ_2^i	θ_3^i
1	17	1.0031	1.1219	1.0256	17	1.0049	1.1219	1.0493
2	44	0.9213	0.9727	1.0410	44	0.9352	1.0084	0.9249
3	61	0.9482	0.9671	0.9661	61	0.9997	0.9427	0.9404
4	87	0.8859	0.9974	0.9653	87	0.7389	1.0452	0.9905
5	157	1.0435	0.9350	1.0432	157	1.0933	1.0019	1.0049
6	200	1.0309	1.0199	1.0382	200	0.9826	1.0210	0.9794
7	422	1.0270	0.9775	0.9669	422	0.9706	0.9939	0.9917
8	464	0.9581	1.0462	1.0523	464	1.0128	1.0508	1.0324
9	483	0.9765	0.9787	1.0291	483	1.0382	1.0328	1.0246
10	502	1.0532	1.0737	1.0136	502	1.0359	1.0073	1.0162
11	722	0.9812	0.9959	0.9919	722	0.9700	0.9695	1.0011
12	754	1.0003	1.0009	0.9189	754	1.0137	0.9987	1.3481
13	870	1.0579	0.9921	1.1378	870	1.0328	1.0033	1.1420
14	930	1.0275	0.9907	0.9978	930	0.9995	0.9812	1.1848
15	1113	1.0009	0.9960	1.0706	1113	0.9304	0.9801	0.9897

The following table gives the drift coefficient's parameter estimates $u_1(\hat{m}_k, k)$, $\kappa_{1,1}(\hat{m}_k, k)$, $\kappa_{1,2}(\hat{m}_k, k)$, $\kappa_{1,3}(\hat{m}_k, k)$, $u_2(\hat{m}_k, k)$, $\kappa_{2,1}(\hat{m}_k, k)$, $\kappa_{2,2}(\hat{m}_k, k)$, $\kappa_{2,3}(\hat{m}_k, k)$, $u_3(\hat{m}_k, k)$, $\kappa_{3,1}(\hat{m}_k, k)$, $\kappa_{3,2}(\hat{m}_k, k)$, $\kappa_{3,3}(\hat{m}_k, k)$ for the decoupled dynamical system for **y** with jump.

Table 22: Estimates $\hat{m}_k$, $u_1(\hat{m}_k,k)$, $\kappa_{1,1}(\hat{m}_k,k)$, $\kappa_{1,2}(\hat{m}_k,k)$, $\kappa_{1,3}(\hat{m}_k,k)$, $u_2(\hat{m}_k,k)$, $\kappa_{2,1}(\hat{m}_k,k)$, $\kappa_{2,2}(\hat{m}_k,k)$, $\kappa_{2,3}(\hat{m}_k,k)$, $u_3(\hat{m}_k,k)$, $\kappa_{3,1}(\hat{m}_k,k)$, $\kappa_{3,2}(\hat{m}_k,k)$, $\kappa_{3,3}(\hat{m}_k,k)$ (with jump).

t_k		Natural gas				Crude oil				Coal			
	$\hat{m}_k$	u_1	$\kappa_{1,1}$	$\kappa_{1,2}$	$\kappa_{1,3}$	u_2	$\kappa_{2,1}(\hat{m}_k,k)$	$\kappa_{2,2}$	$\kappa_{2,3}$	$u_3(\hat{m}_k,k)$	$\kappa_{3,1}$	$\kappa_{3,2}$	$\kappa_{3,3}$
				$\times 10^{-16}$	$\times 10^{-16}$		$\times 10^{-16}$		$\times 10^{-16}$		$\times 10^{-16}$	$\times 10^{-16}$	
11	1	4.1593	0	0	0	57.7000	0	0	0	16.7407	0	0	0
12	4	4.2000	0	0	0	58.6313	0.0011	0.0310	-0.0012	16.2395	0	0	-0.0376
13	6	4.0616	0.0679	-0.0054	-0.0035	58.5378	-0.0035	0.0205	0.0032	16.2680	0	0	0.1069
14	2	4.0616	-0.0242	-0.0179	-0.0035	61.4809	0.0020	0.0098	0	15.5249	0	0	-0.0294
15	7	4.0910	0.6416	-0.2898	0.0078	22.5758	-0.0057	0.0085	0.0012	18.7073	0.0009	-0.0021	0.0318
16	4	4.0160	0	0	0.0078	59.6867	-0.0051	0.0080	0	17.0060	0	-0.0021	0
17	2	4.9575	0	0	0.0078	60.3710	-0.0005	0.0207	0	12.8918	-0.0005	-0.0002	0.0318
18	8	4.9575	-0.1947	0	0.0078	62.3437	0.0005	-0.0008	0	16.5954	0.0002	0.0008	0.0662
19	4	3.3190	-0.4472	0.6760	0.0078	74.6911	-0.0008	-0.0019	0	17.9932	0	-0.0002	0
20	1	3.4762	-0.2540	-0.0048	0.0078	65.9190	0.0026	-0.0006	1	17.6485	0	0	0.0499
...	...	...	...	...	...	...	...	...	...	...	...	...	...
494	1	4.1457	0	0	-0.0001	115.1875	0.0002	0.0053	0	33.3359	0.0003	-0.0003	0.0326
495	1	4.2877	0.1184	-0.0014	0	124.5218	0.0008	0.0056	0	30.1732	0.0002	-0	0.0412
496	1	4.2238	0.2582	0.0011	0.0003	106.8349	0.0003	0.0113	-0.0006	34.9907	0.0034	0	0.0097
497	5	4.0998	0.0477	-0.0006	-0.0002	108.4725	-0.0003	0.0162	-0.0033	33.3388	0.0002	0	0.0443
498	8	4.0592	0.0201	0.0010	0	104.8926	0	0	0.0003	35.1174	0	0.0001	0.0207
499	1	4.3433	0.2118	-0.0014	0	109.2551	-0.0002	0.0048	0.0003	33.2862	0.0010	0.0001	0.0068
500	4	2.4519	0	0	0	111.7067	0	0	0	36.1647	0.0003	0.0003	0.0079
501	1	4.2415	0.4108	0	-0.0015	110.7517	-0.0006	0.0026	0.0009	34.9145	-0.0001	-0.0004	0.0407
502	2	4.3633	0.3210	0.0002	-0.0001	103.6326	-0.0019	0.0023	-0.0002	34.8337	-0.0001	-0.0001	0.0140
503	2	4.2911	0.1276	0.0003	0.0043	112.1547	-0.0033	0.0030	0.0027	35.8389	0.0005	0.0001	0.0211
504	7	4.5942	-0.0125	-0.0002	-0.0031	111.1278	0.0010	0.0072	0.0006	33.6875	-0.0021	0	0.0268
505	2	3.1882	0.0666	0.0009	0	106.1919	-0.0009	0.0110	0.0011	33.6640	-0.0011	-0.0002	0.0231
...	...	...	...	...	...	...	...	...	...	...	...	...	...
1102	1	3.5909	0	0	0.0008	110.3777	0.0006	0.0045	0	5.1761	0.0067	-0.0029	-0.0044
1103	6	3.5303	0.1166	0.0002	-0.0003	111.1585	-0.0003	0.0083	0	5.4558	-0.0019	0.0014	0.0600
1104	4	3.5314	0.0809	0.0018	0	109.0996	-0.0007	0.0095	0.0013	4.8000	0.0005	0.0006	0.1742
1105	1	3.7100	0.2234	-0.0013	-0.0015	106.5667	0.0033	0.0073	-0.0020	5.4226	-0.0082	0.0020	0.0932
1106	8	3.4084	0.1098	0.0001	0	106.5989	0.0003	0.0030	0.0023	5.3360	-0.0023	0.0005	0.0956
1107	7	3.5520	0.1086	0.0001	-0.0070	103.4473	-0.0020	0.0037	-0.0045	4.3586	-0.0005	0.0004	0.1418
1108	5	3.9233	0.0601	0.0007	0	102.8550	0	0.0040	0	4.6582	-0.0010	0	0.1388
1109	8	3.5328	0.0417	0	0	103	-0.0002	0.0089	-0.0005	4.9663	-0.0019	0.0008	0.1279
1110	5	3.8399	0.0212	0.0004	0	102.8800	0	0	0	4.7286	-0.0037	-0.0030	0.0740

The following figures show the parameter estimates $u_1(\hat{m}_k,k)$, $u_2(\hat{m}_k,k)$, and $u_3(\hat{m}_k,k)$ for the decoupled dynamical system for $\mathbf{y}$ with jump.

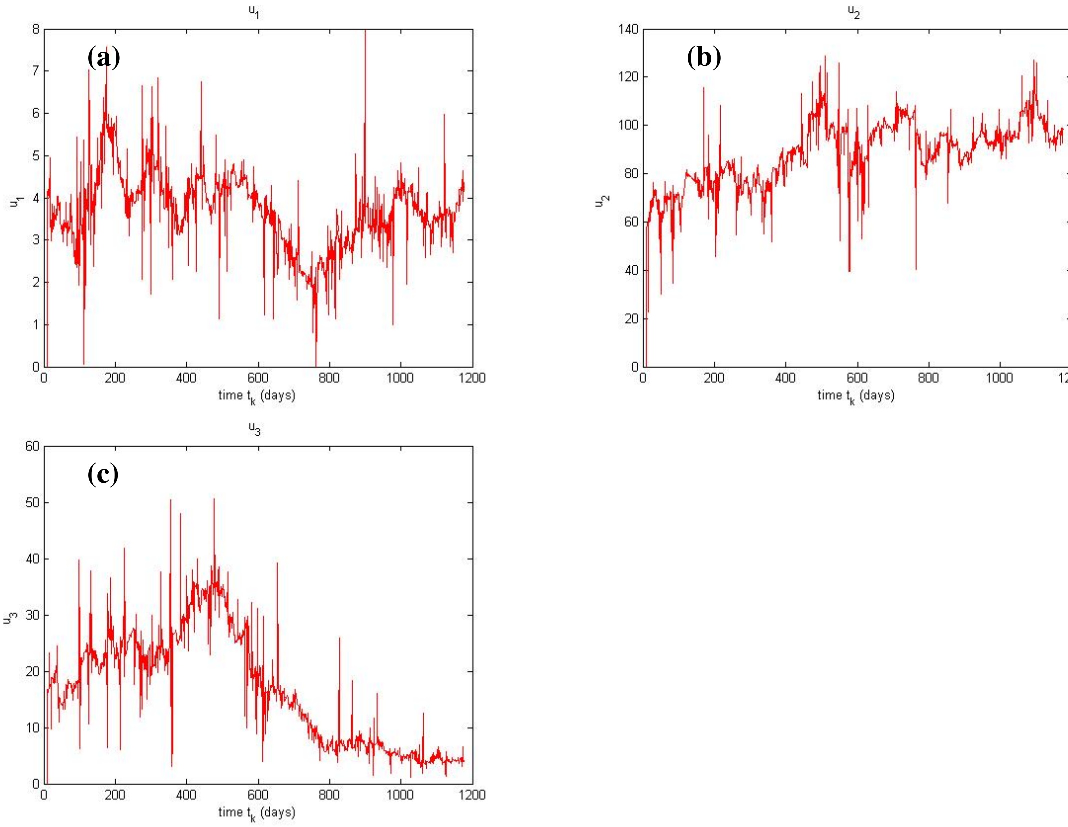
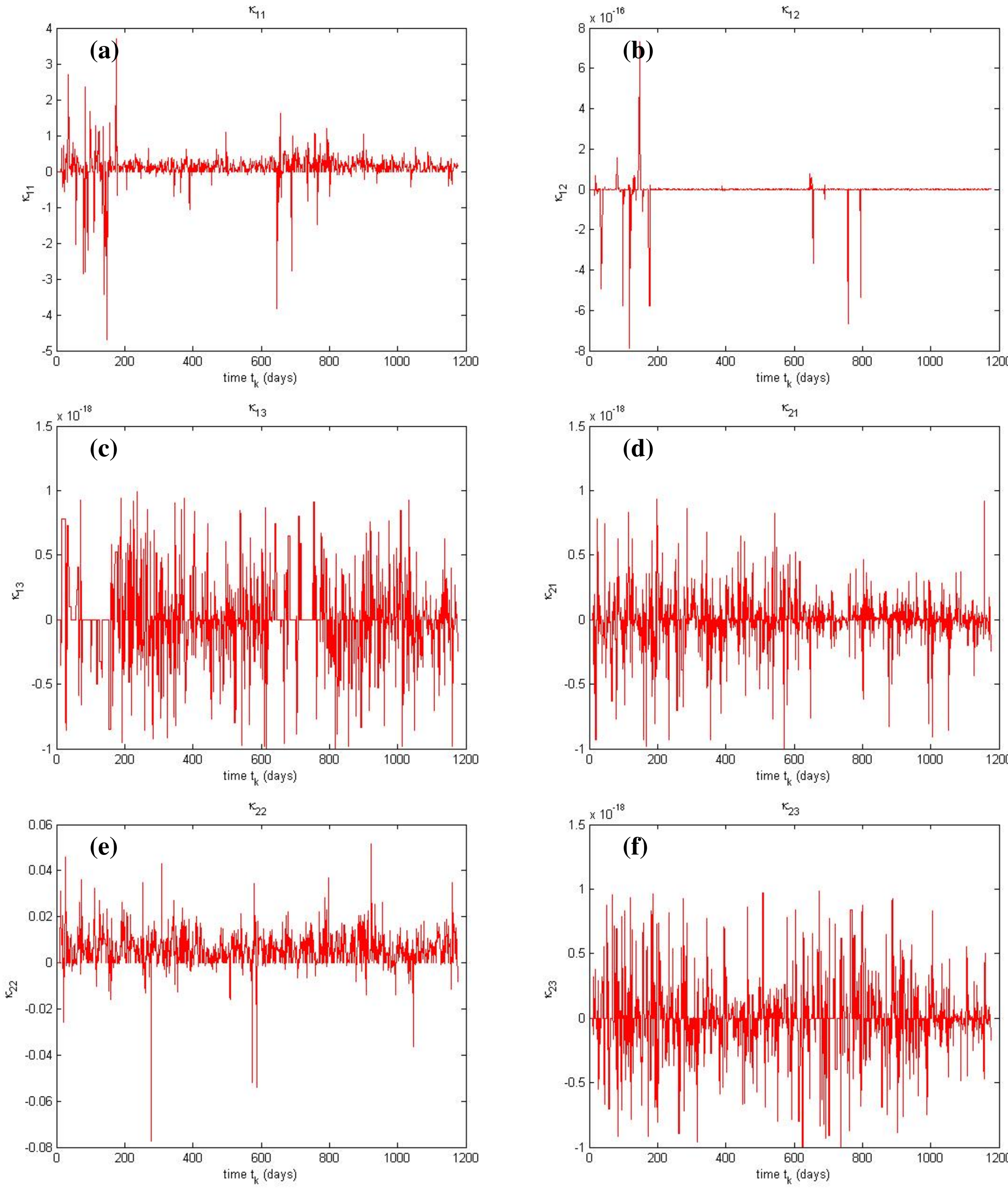


Figure 32.: The graph of mean level $u_1(\hat{m}_k, k)$, $u_2(\hat{m}_k, k)$ and $u_3(\hat{m}_k, k)$ for Natural gas, Crude oil and Coal, respectively (with jump).

Figures 32: **(a), (b) and (c)** are the graphs of $u_1(\hat{m}_k, k)$, $u_2(\hat{m}_k, k)$, and $u_3(\hat{m}_k, k)$ against time t_k for the daily Henry Hub natural gas price data set [27], daily crude oil price data set [28], and daily coal price data set, respectively. By plotting the real data sets (shown in Figure 38, it is easily seen that the graphs of $u_1(\hat{m}_k, k)$, $u_2(\hat{m}_k, k)$ and $u_3(\hat{m}_k, k)$ are similar to the graph of the Henry Hub Natural gas, Crude Oil, and Coal data set, respectively. We expect this to happen because u_j, $j \in I(1, 3)$ are the equilibrium spot price processes described in (9.3).

The graph of the interaction parameters $\kappa_{1,1}(\hat{m}_k, k)$, $\kappa_{1,2}(\hat{m}_k, k)$, $\kappa_{1,3}(\hat{m}_k, k)$, $\kappa_{2,1}(\hat{m}_k, k)$, $\kappa_{2,2}(\hat{m}_k, k)$, $\kappa_{2,3}(\hat{m}_k, k)$, $\kappa_{3,1}(\hat{m}_k, k)$, $\kappa_{3,2}(\hat{m}_k, k)$, and $\kappa_{3,3}(\hat{m}_k, k)$ for the decoupled dynamical system for **y** with jump and estimates in Table 22 are given below:



(a)
(b)
(c)
(d)
(e)
(f)

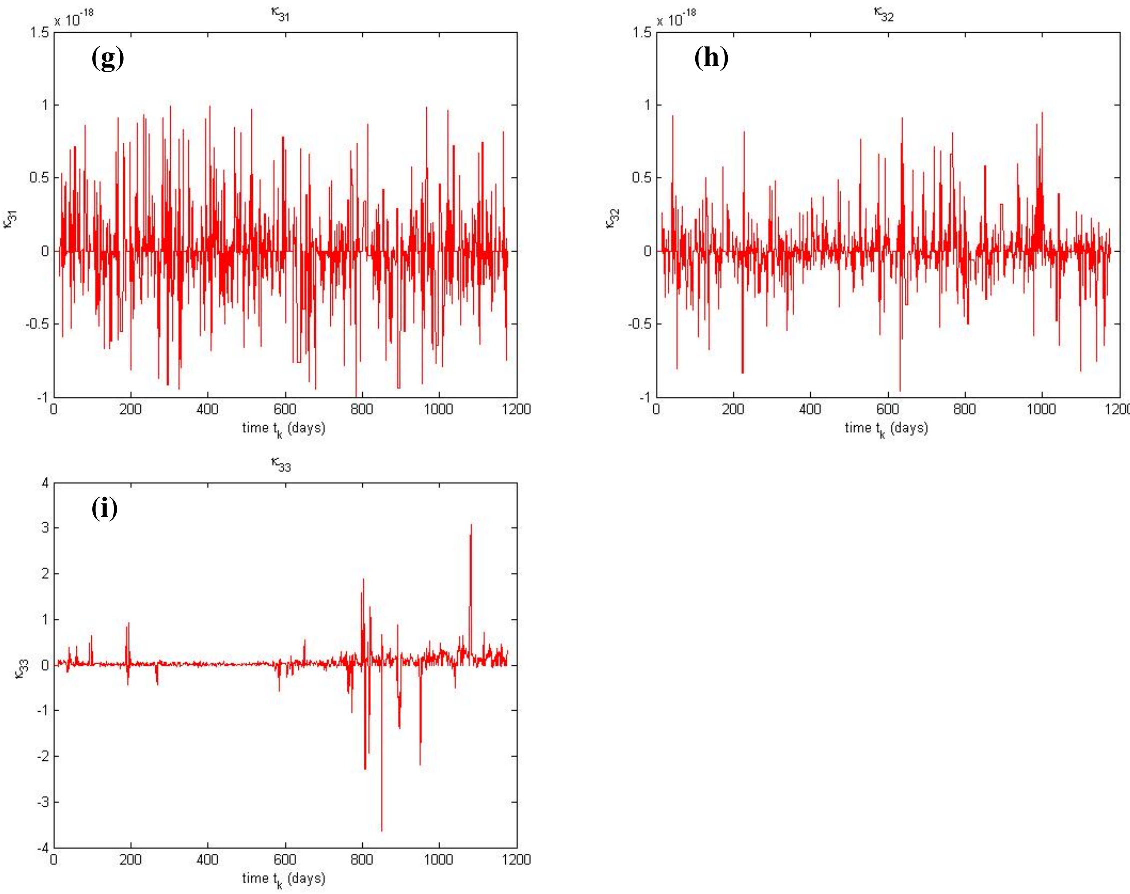


Figure 33.: The graph of interaction coefficients $\kappa_{1,1}(\hat{m}_k, k)$, $\kappa_{1,2}(\hat{m}_k, k)$, $\kappa_{1,3}(\hat{m}_k, k)$, $\kappa_{2,1}(\hat{m}_k, k)$, $\kappa_{2,2}(\hat{m}_k, k)$, $\kappa_{2,3}(\hat{m}_k, k)$, $\kappa_{3,1}(\hat{m}_k, k)$, $\kappa_{3,2}(\hat{m}_k, k)$, $\kappa_{3,3}(\hat{m}_k, k)$ (with jump).

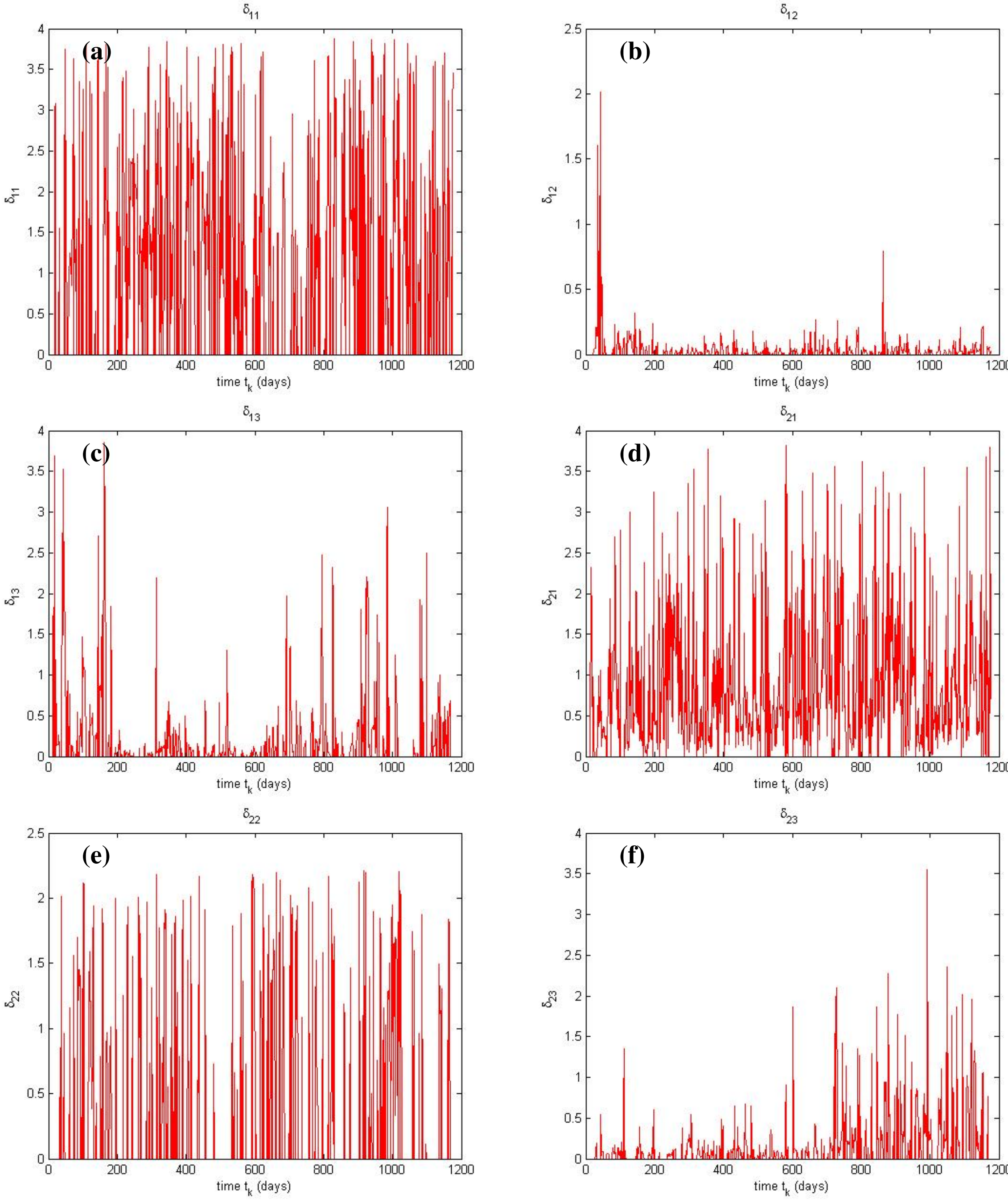
Figures 33 $(a)-(i)$ show the graph of the ϵ- sub-optimal interaction coefficient parameters $\kappa_{1,1}(\hat{m}_k, k)$, $\kappa_{1,2}(\hat{m}_k, k)$, $\kappa_{1,3}(\hat{m}_k, k)$, $\kappa_{2,1}(\hat{m}_k, k)$, $\kappa_{2,2}(\hat{m}_k, k)$, $\kappa_{2,3}(\hat{m}_k, k)$, $\kappa_{3,1}(\hat{m}_k, k)$, $\kappa_{3,2}(\hat{m}_k, k)$, $\kappa_{3,3}(\hat{m}_k, k)$. The interaction coefficients $\kappa_{j,l}$, $j \neq l$ are negligible, because each estimate is $<< 10^{-15}$. Thus, this shows that the model describing the mean spot price, y_j, is mainly characterized by the market potential $\kappa_{j,j}^{i-1}\left(u_j^{i-1} - y_j\right) y_j$, $j \in I(1, n)$, $i \in I(1, K^*)$.

The table below shows the estimates of the diffusion coefficient's parameters for **y**.

Table 23: Estimates $\delta_{1,1}(\hat{m}_k, k)$, $\delta_{1,2}(\hat{m}_k, k)$, $\delta_{1,3}(\hat{m}_k, k)$, $\delta_{2,1}(\hat{m}_k, k)$, $\delta_{2,2}(\hat{m}_k, k)$, $\delta_{2,3}(\hat{m}_k, k)$, $\delta_{3,1}(\hat{m}_k, k)$, $\delta_{3,2}(\hat{m}_k, k)$, $\delta_{3,3}(\hat{m}_k, k)$ (with jump).

t_k	Natural gas			Crude oil			Coal		
	$\delta_{1,1}$	$\delta_{1,2}$	$\delta_{1,3}$	$\delta_{2,1}$	$\delta_{2,2}$	$\delta_{2,3}$	$\delta_{3,1}$	$\delta_{3,2}$	$\delta_{3,3}$
11	0.0062	0.0010	0.0001	1.5277	0.0078	0.0011	0	0	0.0218
12	0.0182	0.9002	0	1.6227	0.0010	0	0	0	0.0988
13	0.0239	0.0802	1.7280	0	1.7694	0	0.6374	0.6374	0.0959
14	0	0.0001	0.6027	2.3258	0	0	1.6564	1.6564	0.0847
15	0	0.8001	0.6210	2.3252	0	0	1.6650	1.6650	0.0111
16	0.0455	0.0007	3.6877	2.3217	0	1.2215	1.6724	1.6724	0
17	0	0.9876	0	1.6425	0	0	1.7719	1.7719	0
18	3.0410	0.9351	0	1.3105	0	0.1070	1.7630	1.7630	0.0434
19	2.7713	0.6680	0	1.1052	0	0	1.7400	1.7400	0
20	2.8461	1.7795	0	0.1196	0	0.0983	0	0.4555	0
...	...	...	...	...	...	...	...	...	...
495	1.1229	0	0.0584	0.5488	0.1104	0.0761	0	0	1.3987
496	0.6946	0	0.6613	0.5767	0.0715	0.0610	0	0	1.3017
497	1.1229	0.0095	0.0988	0.6499	0.0870	0.0633	1.1317	1.1317	1.3069
498	0.6946	0.0101	0	0	0	0.0320	1.0294	1.0294	1.5410
499	0.7353	0.0066	0.0384	0	0.0922	0.0330	0.7317	0.7317	1.2225
500	1.7509	0.0069	0.0283	0.4307	0.4545	0.0413	0.4826	0.4826	1.2254
501	2.1299	0.0077	0.0282	0.5043	0.7873	0.0308	0.4272	0.4272	1.5587
502	0.9778	0.0077	0	0.2878	0	0	0.5239	0.5239	1.8713
503	0.9872	0	0	0.2909	0	0	1.4523	1.4523	1.8874
504	1.1329	0	0	0.3707	0.4261	0	0	0	0
505	1.9178	0	0	0.3812	0.7292	0.1724	0	0	0
...	...	...	...	...	...	...	...	...	...
1102	0	0.0331	0.7183	0.9297	0.0434	0.0680	0	0	1.1355
1103	1.5077	0.0626	0.2048	1.1017	0.0421	0.1510	0	0	1.4133
1104	0.4444	0.0435	0.4622	0.1939	0.1078	0	0.0814	0	1.1672
1105	3.5933	0	0.3646	0.1922	0	0.7273	0.2726	0.2726	1.3023
1106	2.4964	0	0.3919	0	0.0684	1.0179	0.3296	0.3296	1.4111
1107	2.4600	0	0.8995	0.2001	0.1510	0.9354	0	0	1.7245
1108	2.0262	0	0.6325	0.3781	0.0814	0.8825	0.1878	0.1878	1.0915
1109	1.7828	0	0.6116	0.4024	0.0332	0.8812	0	0	1.3191
1110	1.2706	0	0.1001	0.3252	0.0155	0.8078	0	0	1.0233

The graph of the diffusion coefficient's parameter for the decoupled dynamical system for **y** with jump are given below:


δ11
(a)
δ11
time tk (days)
δ12
(b)
δ12
time tk (days)
δ13
(c)
δ13
time tk (days)
δ21
(d)
δ21
time tk (days)
δ22
(e)
δ22
time tk (days)
δ23
(f)
δ23
time tk (days)

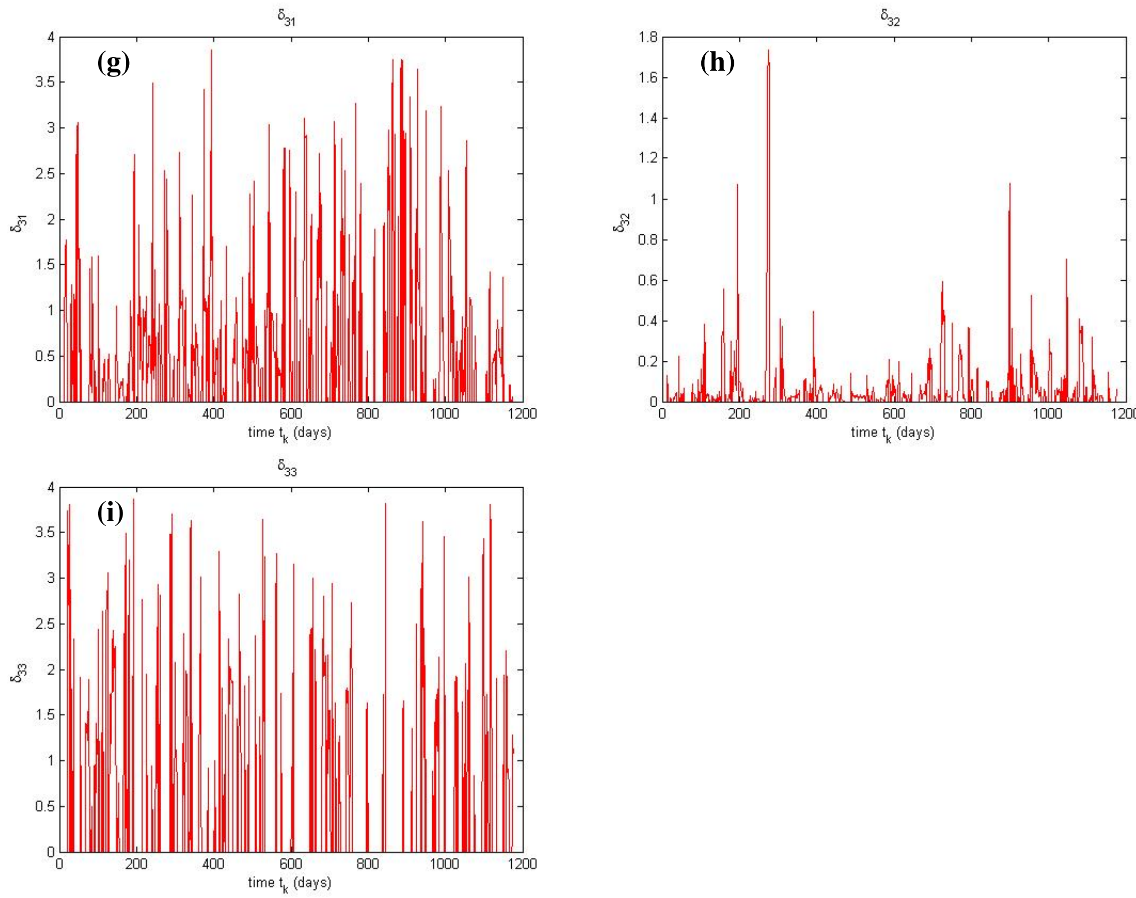


Figure 34.: The graph of interaction coefficients $\delta_{1,1}(\hat{m}_k,k)$, $\delta_{1,2}(\hat{m}_k,k)$, $\delta_{1,3}(\hat{m}_k,k)$, $\delta_{2,1}(\hat{m}_k,k)$, $\delta_{2,2}(\hat{m}_k,k)$, $\delta_{2,3}(\hat{m}_k,k)$, $\delta_{3,1}(\hat{m}_k,k)$, $\delta_{3,2}(\hat{m}_k,k)$, $\delta_{3,3}(\hat{m}_k,k)$ (with jump).

Figures 34 $(a)-(i)$ show the graph of the ϵ- sub-optimal interaction measure of fluctuation coefficient parameters $\delta_{1,1}(\hat{m}_k,k)$, $\delta_{1,2}(\hat{m}_k,k)$, $\delta_{1,3}(\hat{m}_k,k)$, $\delta_{2,1}(\hat{m}_k,k)$, $\delta_{2,2}(\hat{m}_k,k)$, $\delta_{2,3}(\hat{m}_k,k)$, $\delta_{3,1}(\hat{m}_k,k)$, $\delta_{3,2}(\hat{m}_k,k)$, $\delta_{3,3}(\hat{m}_k,k)$.

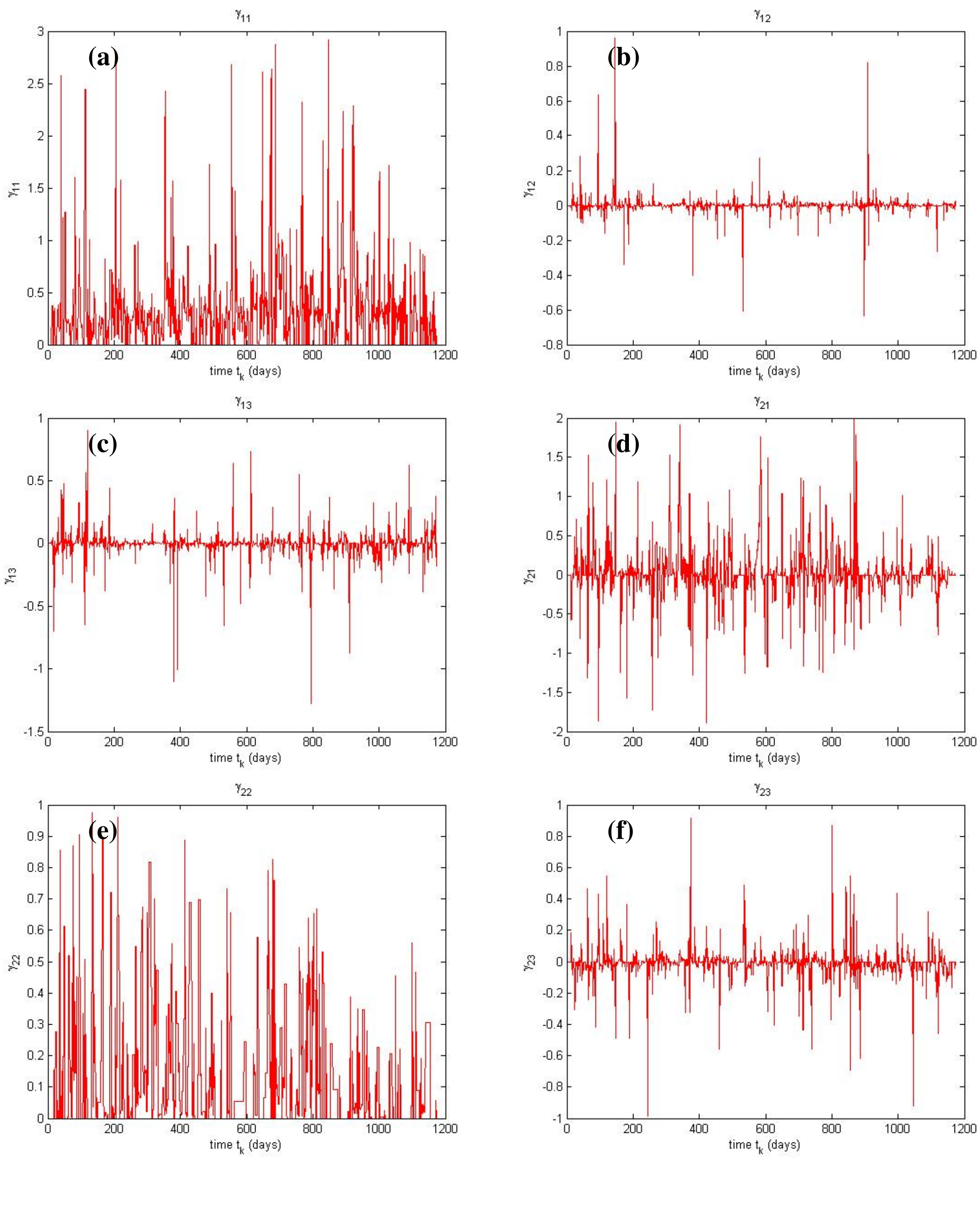
The following table gives the drift coefficient's parameter estimates $\beta_1(\hat{m}_k,k)$, $\gamma_{1,1}(\hat{m}_k,k)$, $\gamma_{1,2}(\hat{m}_k,k)$, $\gamma_{1,3}(\hat{m}_k,k)$, $\beta_2(\hat{m}_k,k)$, $\gamma_{2,1}(\hat{m}_k,k)$, $\gamma_{2,2}(\hat{m}_k,k)$, $\gamma_{2,3}(\hat{m}_k,k)$, $\beta_3(\hat{m}_k,k)$, $\gamma_{3,1}(\hat{m}_k,k)$, $\gamma_{3,2}(\hat{m}_k,k)$, and $\gamma_{3,3}(\hat{m}_k,k)$ for the dynamical system for **p** with jump.

Table 24: Estimates $\beta_1(\hat{m}_k, k)$, $\gamma_{1,1}(\hat{m}_k, k)$, $\gamma_{1,2}(\hat{m}_k, k)$, $\gamma_{1,3}(\hat{m}_k, k)$, $\beta_2(\hat{m}_k, k)$, $\gamma_{2,1}(\hat{m}_k, k)$, $\gamma_{2,2}(\hat{m}_k, k)$, $\gamma_{2,3}(\hat{m}_k, k)$, $\beta_3(\hat{m}_k, k)$, $\gamma_{3,1}(\hat{m}_k, k)$, $\gamma_{3,2}(\hat{m}_k, k)$, $\gamma_{3,3}(\hat{m}_k, k)$ (with jump).

t_k	Natural gas				Crude oil				Coal			
	β_1	$\gamma_{1,1}$	$\gamma_{1,2}$	$\gamma_{1,3}$	β_2	$\gamma_{2,1}(\hat{m}_k, k)$	$\gamma_{2,2}$	$\gamma_{2,3}$	β_3	$\gamma_{3,1}$	$\gamma_{3,2}$	$\gamma_{3,3}$
11	0	0	0	0	0	0	0	0	0	0	0	0
12	0.1681	0.3497	-0.0109	0.0248	-0.4815	0.1626	-0.4066	-0.0123	0.7665	-0.3259	0.0205	-0.0198
13	0.1592	0.3755	-0.0102	0.0228	-0.7778	-0.5752	-0.0578	0.1870	1.0795	-0.2904	0.0135	-0.0217
14	7.3439	0.3755	0.0488	-0.5478	1.7680	-0.5555	-0.2058	0.0291	0.8543	-0.2056	0.0034	-0.0064
15	0.3336	0.3652	-0.0127	0.0213	-0.8999	0.0601	-0.1110	0.0405	0.5144	-0.1264	-0.0062	0.0127
16	0.4709	0.2780	-0.0116	0.0104	-0.8999	0.0601	-0.1110	-0.0292	0.0017	-0.0002	0	0
17	0.3277	0.2780	0.0768	-0.2633	1.3349	0.0027	-0.0330	-0.0750	-0.6285	-0.0569	0.0016	0.0262
18	0.3277	1.3156	-0.1491	-0.1646	1.5419	-0.0088	0.1205	-0.0892	-1.3275	-0.0703	0.0080	0.0386
19	0	0	0	0	-0.1785	-0.0368	-0.0062	0.0189	-0.8091	-0.0001	-0.0083	0.0466
20	0.6985	0.4990	-0.0069	-0.0187	-0.1513	-0.0778	-0.0096	0.0255	-0.0182	0.0080	0.0001	0.0003
...	....	...	...	...	...	...	...	...	...	...	...	...
495	0.1201	0.2264	0.0007	-0.0056	0.2756	0.1131	-0.9587	-0.0087	-0.0288	-0.0780	-0.0019	0.0136
496	0.1809	0.2085	0.0009	-0.0082	0.2898	-0.0431	0.7329	-0.0133	0.1324	-0.1434	-0.0018	0.0158
497	0.2442	0.1597	0.0007	-0.0093	3.1030	-0.0495	-0.0462	-0.0862	0.9772	-0.2426	-0.0065	0.0177
498	0.2742	0.2651	0.0020	-0.0145	1.3147	0.0148	-0.0009	-0.0411	0.2770	-0.1888	-0.0037	0.0217
499	0.3320	0.3298	-0.0009	-0.0070	0.8430	0.0070	0.0283	-0.0265	0.0931	-0.1551	-0.0041	0.0237
500	0.5035	0.2337	-0.0007	-0.0128	1.3320	0.3949	-0.3308	-0.0838	0.4175	-0.2331	-0.0034	0.0222
501	0.6328	0.2612	-0.0034	-0.0077	0.3251	-0.1148	0.0548	0.0042	0.7896	-0.2546	-0.0093	0.0305
502	0.5403	0.2457	-0.0014	-0.0113	0.4863	-0.1315	0.0343	0.0020	3.7990	0.0420	-0.0524	0.0421
503	0.4794	0.2098	-0.0028	-0.0050	0.1239	0.0013	0.0202	-0.0040	9.6735	0.0736	-0.1152	0.0648
504	-0.3308	-0.5600	0.0258	-0.0737	0.2867	-0.0391	0.0009	-0.0034	4.2547	0.3506	-0.0669	0.0382
505	1.1680	0.8346	-0.0274	0.0542	0.1198	-0.0588	-0.3412	0.0092	2.2295	0.0897	-0.0357	0.0306
...	...	...	...	...	...	...	...	...	...	...	...	...
1102	0.6765	0.0455	-0.0020	-0.0908	0.4026	-0.2544	0.2045	0.1058	-5.9294	0.6777	0.0292	0.0068
1103	1.1804	0.4214	-0.0149	0.0837	-0.6549	0.1780	0.0070	0.0018	-6.3380	0.7440	0.0291	0.0106
1104	0.1069	0.2489	-0.0009	-0.0014	-2.1178	0.3406	0.1959	0.1826	-3.8701	0.5681	0.0157	0.0021
1105	0.0139	0.2777	-0.0001	-0.0008	0.3958	-0.0274	0.0642	-0.0620	-4.0701	0.1880	0.0091	0.0514
1106	-0.2513	0.4043	0.0031	-0.0164	0.4097	0.0060	0.1536	-0.0907	-5.0668	0.3261	0.0178	0.0419
1107	0.0670	0.3163	-0	-0.0145	0.2906	0.0485	0.2310	-0.0989	-5.0668	0.4016	0.0308	0.1474
1108	1.0112	0.6861	-0.0091	-0.0107	0.4281	0.0048	0.1337	-0.0933	-5.0668	0.4650	0.0304	0.1295
1109	0.5020	0.5370	-0.0030	-0.0375	0.3645	-0.0168	0.1078	-0.0641	-5.0668	0.4156	0.0311	0.1396
1110	0.1420	0.3295	0.0009	-0.0484	0.1728	-0.0189	-0.0164	-0.0230	-6.6650	0.4509	0.0099	0.0831

Table 24 shows the estimates of the parameters $\hat{m}_k$,$\beta_1(\hat{m}_k, k)$, $\gamma_{1,1}(\hat{m}_k, k)$, $\gamma_{1,2}(\hat{m}_k, k)$, $\gamma_{1,3}(\hat{m}_k, k)$, $\hat{m}_k$, $\beta_2(\hat{m}_k, k)$, $\gamma_{2,1}(\hat{m}_k, k)$, $\gamma_{2,2}(\hat{m}_k, k)$, $\gamma_{2,3}(\hat{m}_k, k)$, $\hat{m}_k$,$\beta_3(\hat{m}_k, k)$, $\gamma_{3,1}(\hat{m}_k, k)$, $\gamma_{3,2}(\hat{m}_k, k)$, $\gamma_{3,3}(\hat{m}_k, k)$, for each of the energy commodity data sets. According to (9.47), the estimate $\gamma_{j,l}(\hat{m}_k, k)$, $j \neq l$, is positive if commodity p_l is cooperating with commodity p_j, and negative if commodity p_l is competing with commodity p_j. There is no interaction between the two commodities if $\gamma_{j,l}(\hat{m}_k, k) = 0$. It is apparent from the graph (from $\gamma_{1,3}(\hat{m}_k, k)$ in Column 6) that coal is competing with natural gas during this period because the estimates of $\gamma_{1,3}(\hat{m}_k, k)$ are mostly negative. It is apparent that natural gas and crude oil are either cooperating or competing, depending on the time period.

In the following, the graph of the drift coefficient's parameters with estimates in Table 24 are given below:



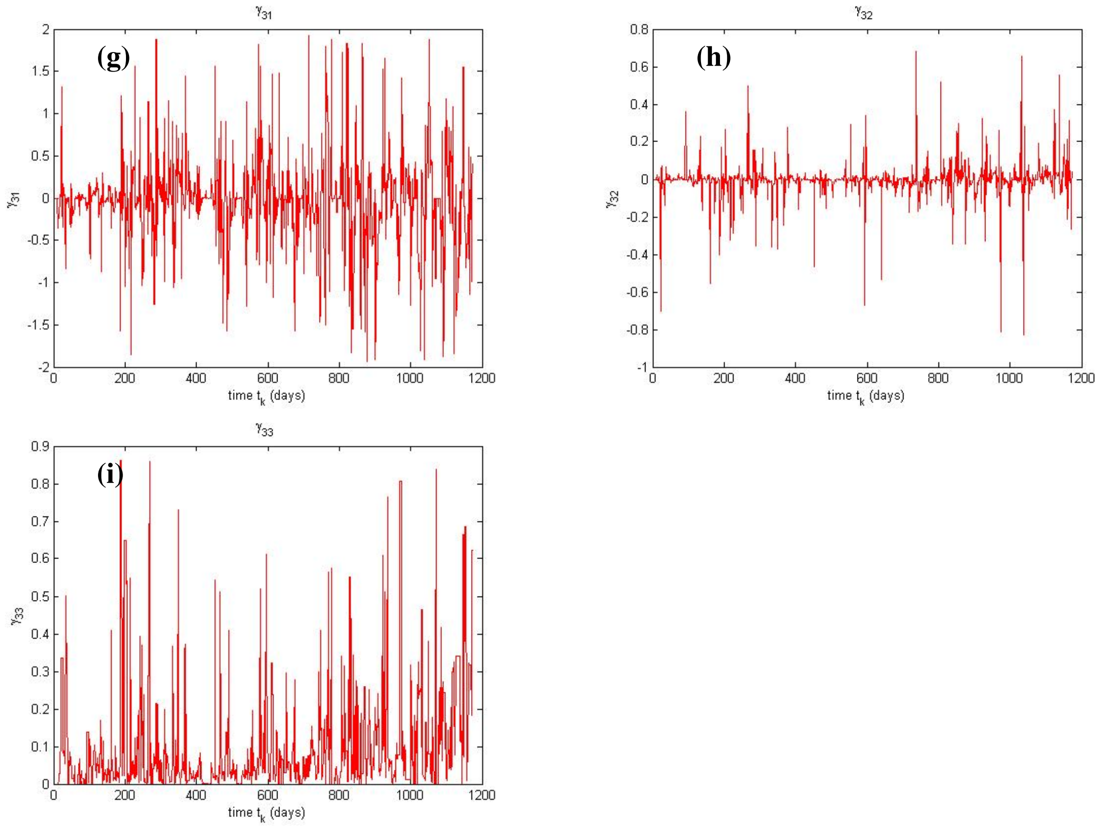


Figure 35.: The graph of interaction coefficients $\gamma_{1,1}(\hat{m}_k, k)$, $\gamma_{1,2}(\hat{m}_k, k)$, $\gamma_{1,3}(\hat{m}_k, k)$, $\gamma_{2,1}(\hat{m}_k, k)$, $\gamma_{2,2}(\hat{m}_k, k)$, $\gamma_{2,3}(\hat{m}_k, k)$, $\gamma_{3,1}(\hat{m}_k, k)$, $\gamma_{3,2}(\hat{m}_k, k)$, $\gamma_{3,3}(\hat{m}_k, k)$ (with jump).

Figures 35 $(a) - (i)$ show the graph of the ϵ- sub-optimal interaction coefficient parameters $\gamma_{1,1}(\hat{m}_k, k)$, $\gamma_{1,2}(\hat{m}_k, k)$, $\gamma_{1,3}(\hat{m}_k, k)$, $\gamma_{2,1}(\hat{m}_k, k)$, $\gamma_{2,2}(\hat{m}_k, k)$, $\gamma_{2,3}(\hat{m}_k, k)$, $\gamma_{3,1}(\hat{m}_k, k)$, $\gamma_{3,2}(\hat{m}_k, k)$, $\gamma_{3,3}(\hat{m}_k, k)$. According to (9.47), the estimate $\gamma_{j,l}(\hat{m}_k, k)$, $j \neq l$, is positive if commodity p_l is cooperating with commodity p_j, and negative if commodity p_l is competing with commodity p_j. There is no interaction between the two commodities if $\gamma_{j,l}(\hat{m}_k, k) = 0$. It is apparent from the graph of $\gamma_{1,3}(\hat{m}_k, k)$ that coal is competing with natural gas because the estimates of $\gamma_{1,3}(\hat{m}_k, k)$ are mostly negative. Also, it is apparent that natural gas and crude oil are either cooperating or competing, depending on the time period.

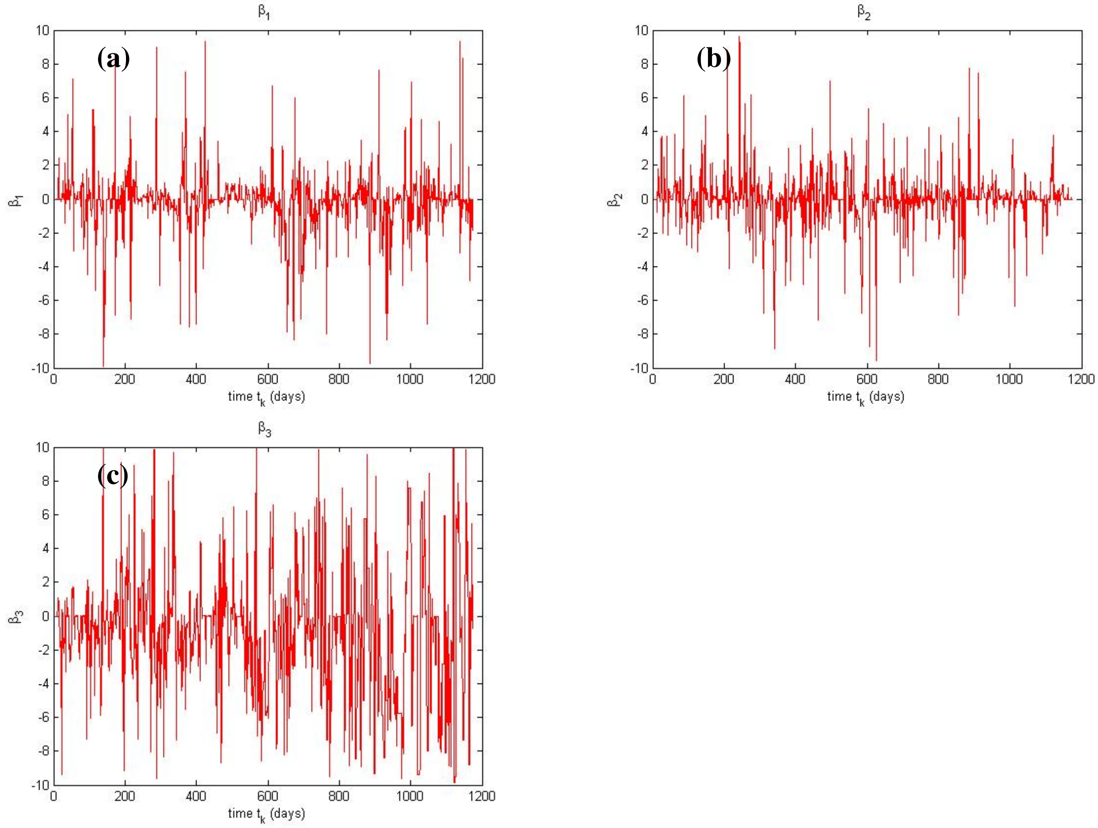


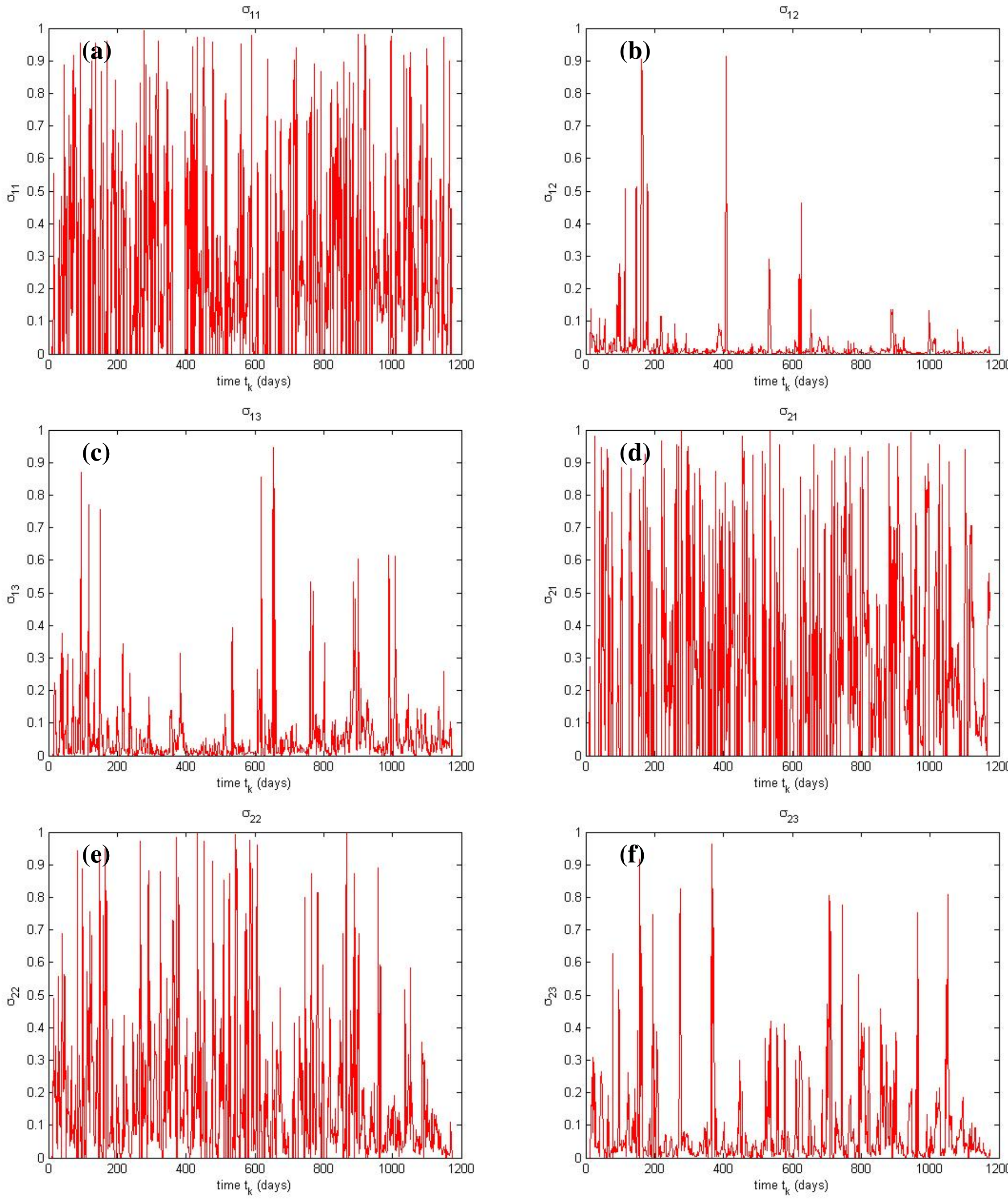
Figure 36.: The graph of mean level $\beta_1(\hat{m}_k, k)$, $\beta_2(\hat{m}_k, k)$ and $\beta_3(\hat{m}_k, k)$ (with jump).

Figures 36: **(a), (b) and (c)** are the graphs of $\beta_1(\hat{m}_k, k)$, $\beta_2(\hat{m}_k, k)$, and $\beta_3(\hat{m}_k, k)$ against time t_k for the daily Henry Hub natural gas price, [27] daily crude oil price [28], and daily coal price data set, respectively.

Table 25: Estimates $\sigma_{1,1}(\hat{m}_k, k)$, $\sigma_{1,2}(\hat{m}_k, k)$, $\sigma_{1,3}(\hat{m}_k, k)$, $\sigma_{2,1}(\hat{m}_k, k)$, $\sigma_{2,2}(\hat{m}_k, k)$, $\sigma_{2,3}(\hat{m}_k, k)$, $\sigma_{3,1}(\hat{m}_k, k)$, $\sigma_{3,2}(\hat{m}_k, k)$, $\sigma_{3,3}(\hat{m}_k, k)$ (with jump).

t_k	Natural gas			Crude oil			Coal		
	$\sigma_{1,1}$	$\sigma_{1,2}$	$\sigma_{1,3}$	$\sigma_{2,1}$	$\sigma_{2,2}$	$\sigma_{2,3}$	$\sigma_{3,1}$	$\sigma_{3,2}$	$\sigma_{3,3}$
11	0	0	0	0	0	0	0	0.1303	0
12	0.0485	0.0004	0.0032	0.2734	0.0166	0	0.0513	0	0.0000
13	0.7333	0	0	0.9445	0	0	0.2489	0	0
14	0.2120	0.1386	0.0133	0.3877	0.4773	0.1195	0.1365	0.0665	0.0086
15	0.4246	0.1318	0.0021	0.03341	0.4894	0.1211	0.0112	0.6107	0.0696
16	0.5538	0.0778	0.1501	0.07751	0.2524	0.0811	0.0651	0.4251	0.0635
17	0.3907	0.0469	0.2230	0.08746	0.1848	0.2463	0	0.4458	0.0478
18	0.3523	0.0180	0.2178	0.04291	0.1877	0.1602	0.5681	0.0592	0.0115
19	0.5116	0.0619	0.2221	0.03266	0.2673	0.2465	0.4999	0.0569	0.0127
20	0.6431	0.0536	0.1866	0.0939	0.1700	0.0781	0.3789	0.3174	0.0046
...	..	...	...	...	...	...	...	...	...
495	0	0.0036	0.0406	0.2286	0.0600	0.0172	0.0110	0.9387	0.0182
496	0.1588	0.0035	0.0107	0.08183	0.3163	0.0102	0	0	0.0016
497	0.1551	0.0009	0.0065	0.07869	0.1453	0.4821	0.7777	0	0.0033
498	0.1576	0.0011	0.0073	0.0120	0.1679	0.3786	0.5334	0	0.0060
499	0.1197	0.0006	0.0059	0.0721	0.2391	0.0172	0.4405	0.1432	0.0097
500	0.3600	0.0001	0.0049	0.0273	0.3960	0.0079	0.6331	0.1410	0.0093
0.5010	0.0514	0.0033	0.0049	0.0182	0.3499	0.0111	0.7690	0.1376	0.0089
0.5020	0.2503	0.0034	0.0042	0.0222	0.1744	0.0132	0.6198	0.1274	0.0066
0.5030	0.1195	0.0147	0.0165	0	0.4283	0.0060	0	0.1530	0.0049
0.5040	0.0974	0.0144	0.0027	0	0.2241	0.0048	0.4778	0.0574	0.0043
0.5050	0.1422	0.0060	0.0131	0.0085	0.2023	0.0054	0.5604	0.0669	0.0004
...	...	...	...	...	...	...	...	...	...
1102	0.1898	0.0016	0.0413	0.8313	0.0767	0.0381	0.6875	0	0.1451
1103	0.2094	0.0015	0.0352	0.8262	0.0673	0.0451	0.7298	0.2808	0.0147
1104	0.1711	0.0011	0.0040	0.6648	0.0915	0.0462	0.5563	0.1831	0.0105
1105	0.1816	0.0012	0.0116	0.6658	0.1049	0.0371	0.6591	0.2874	0.0057
1106	0.1191	0.0011	0.0116	0.6260	0.1155	0.0393	0	0.0196	0.0060
1107	0.0417	0.0012	0.0041	0.4992	0.0781	0.0382	0.0271	0.2559	0.0065
1108	0.1058	0.0033	0.0045	0.0019	0.0589	0.0421	0.6209	0.8289	0.0018
1109	0.1740	0.0021	0	0.0305	0.0446	0.0316	0.8431	0.2366	0.4511
1110	0.2912	0.0021	0.0163	0.0385	0.0342	0.0037	0.2910	0.0489	0.0257

Table 25 gives the ϵ-sub-optimal estimates of the parameters $u_1(\hat{m}_k, k)$, $\sigma_{1,1}(\hat{m}_k, k)$, $\sigma_{1,2}(\hat{m}_k, k)$, $\sigma_{1,3}(\hat{m}_k, k)$, $u_2(\hat{m}_k, k)$, $\sigma_{2,1}(\hat{m}_k, k)$, $\sigma_{2,2}(\hat{m}_k, k)$, $\sigma_{2,3}(\hat{m}_k, k)$, $u_3(\hat{m}_k, k)$, $\sigma_{3,1}(\hat{m}_k, k)$, $\sigma_{3,2}(\hat{m}_k, k)$, $\sigma_{3,3}(\hat{m}_k, k)$ for each of the energy commodity data sets.



σ11
(a)
σ12
(b)
σ13
(c)
σ21
(d)
σ22
(e)
σ23
(f)
time t_k (days)

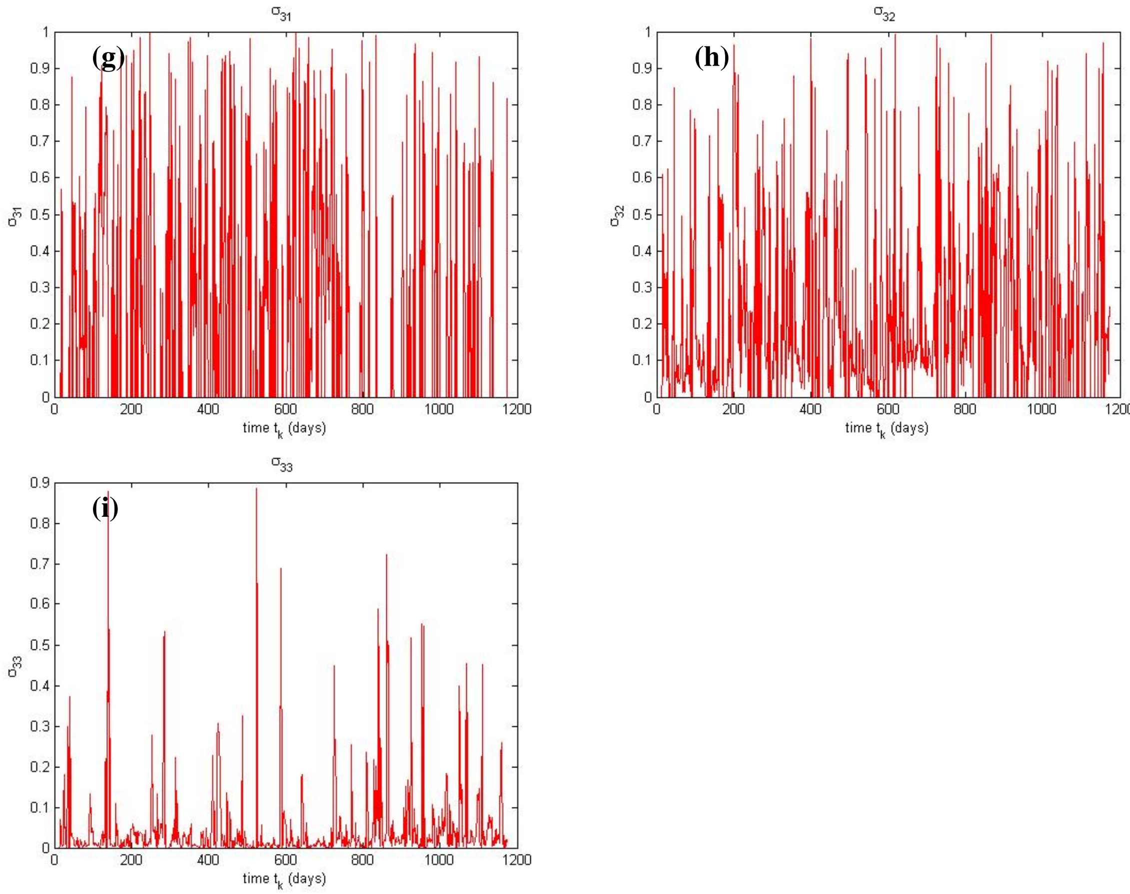


Figure 37.: The graph of $\sigma_{1,1}(\hat{m}_k, k)$, $\sigma_{1,2}(\hat{m}_k, k)$, $\sigma_{1,3}(\hat{m}_k, k)$, $\sigma_{2,1}(\hat{m}_k, k)$, $\sigma_{2,2}(\hat{m}_k, k)$, $\sigma_{2,3}(\hat{m}_k, k)$, $\sigma_{3,1}(\hat{m}_k, k)$, $\sigma_{3,2}(\hat{m}_k, k)$, $\sigma_{3,3}(\hat{m}_k, k)$ for Natural gas, Crude oil and Coal, respectively (with jump).

Figures 37: **(a), (b) and (c)** are the graphs of $\sigma_{1,1}(\hat{m}_k, k)$, $\sigma_{1,2}(\hat{m}_k, k)$, $\sigma_{1,3}(\hat{m}_k, k)$, $\sigma_{2,1}(\hat{m}_k, k)$, $\sigma_{2,2}(\hat{m}_k, k)$, $\sigma_{2,3}(\hat{m}_k, k)$, $\sigma_{3,1}(\hat{m}_k, k)$, $\sigma_{3,2}(\hat{m}_k, k)$, $\sigma_{3,3}(\hat{m}_k, k)$ against time t_k for the daily Henry Hub natural gas price data set [27], daily crude oil price data set [28], and daily coal price data set, respectively.

Table 26: Real and simulated estimates (with jump) for Natural gas, Crude oil, and Coal.

t_k	Natural gas		Crude oil		Coal	
	Real	Simulated p_1^s	Real	Simulated p_2^s	Real	Simulated p_3^s
11	4.0200	4.0500	58.9900	58.5200	16.5900	16.6000
12	3.9900	3.9600	59.5200	59.5099	17.4600	17.4635
13	3.7500	3.6690	61.4500	60.3377	17.8900	17.8886
14	3.7700	3.7341	60.4900	59.4191	17.5500	17.5188
15	3.4100	3.3967	61.1500	60.1580	17.4100	17.4188
16	3.3500	3.3947	62.4800	62.5028	16.7500	16.7789
17	3.4900	3.3900	63.4100	63.5524	17.6600	17.5341
18	3.5500	3.4957	65.0900	64.9224	17.5200	17.8130
19	3.9200	3.8743	66.3100	66.8239	18.5000	18.8453
20	3.8600	3.8241	68.5900	68.1206	19.0600	18.9453
...	...	...	...	...	...	...
494	4.2300	4.2295	106.7000	106.6374	33.2200	33.1852
495	4.1900	4.2191	107.1800	106.9973	32.7600	32.6677
496	4.3300	4.3414	110.8400	111.0084	33.6500	33.8070
497	4.3300	4.3261	111.7200	111.7084	33.7100	33.7061
498	4.3700	4.3863	111.6800	111.2084	34.7500	35.7057
499	4.3200	4.2167	111.7200	111.6126	34.5400	33.5457
500	4.3500	4.3577	112.3100	112.4358	34.0400	34.2862
501	4.3800	4.3535	112.3800	112.5682	33.1000	33.1330
502	4.5100	4.4389	113.3900	113.2925	33.6700	33.6216
503	4.6000	4.6139	113.0300	113.3077	33.9400	33.9216
504	4.6000	4.5964	110.6000	110.8350	33.8300	33.9216
505	4.5900	4.5564	108.7900	108.7598	32.0200	31.9867
...	...	...	...	...	...	...
1102	3.7200	3.6616	108.2300	108.2354	4.7700	4.5482
1103	3.7300	3.6477	106.2600	106.1451	5.0100	5.1120
1104	3.6800	3.6748	104.7000	104.6723	4.9800	5.1180
1105	3.6600	3.6861	103.6200	104.6723	4.7300	4.7893
1106	3.5900	3.6436	103.2200	102.9765	4.6800	4.7074
1107	3.5200	3.5213	102.6800	102.7652	4.6300	4.6419
1108	3.4900	3.4564	103.1000	102.9765	4.7400	4.4016
1109	3.5100	3.2596	102.8600	102.9652	4.3300	4.1826
1110	3.4800	3.4604	102.3600	102.4345	4.1800	4.4606

Table 26 shows the Real and simulated estimates for the spot price processes $p_j(t)$, $j \in I(1, n)$.

The next figure shows the graph of the real and simulated prices (with jump) for Natural gas, Crude oil, and Coal data set

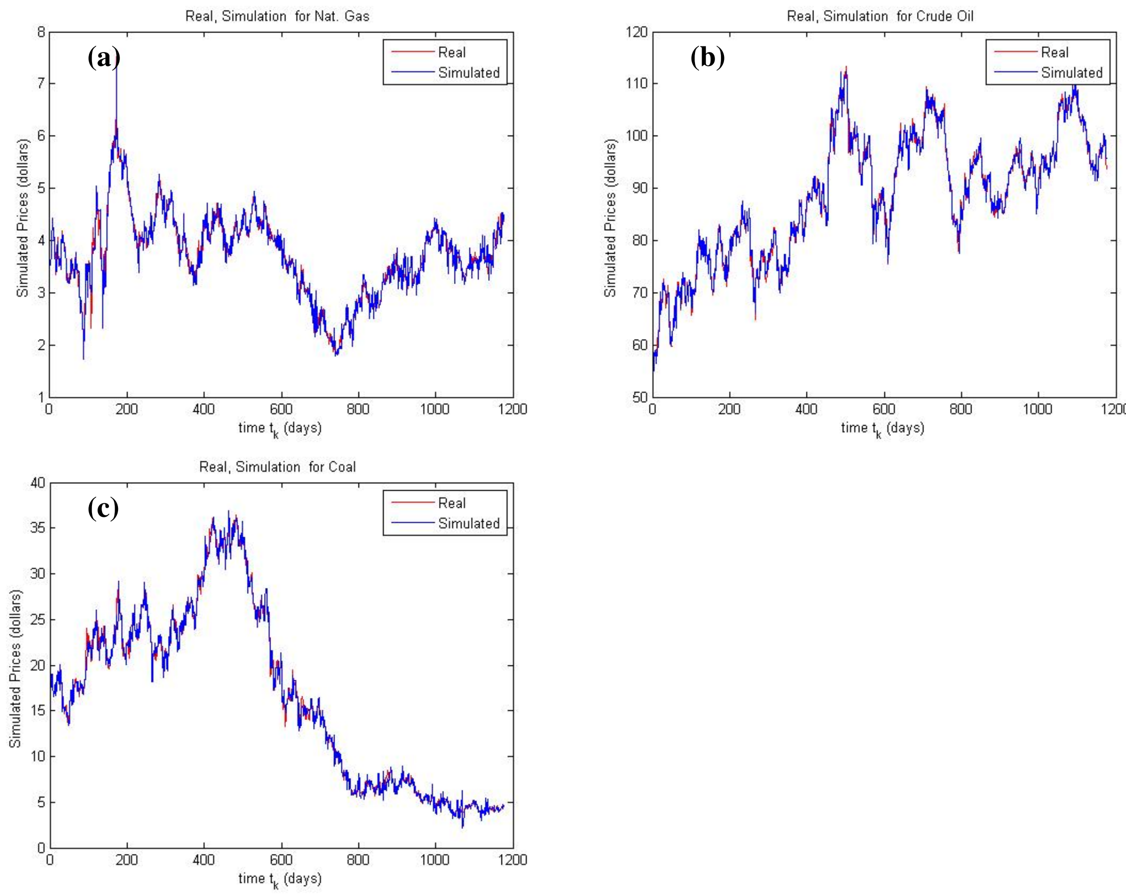


Figure 38.: Real and Simulated Prices (with jump) for Natural gas, Crude oil, and Coal.

Figures 38: **(a), (b), and (c)** show the graph of the Real and Simulated Spot Prices for the daily Henry Hub natural gas data set [27], daily crude oil data set [28], and daily coal data set [26], respectively. The red line represents the real data set $\mathbf{p}$, while the blue line represent the simulated data set $\mathbf{p}^s_{\hat{m}_k,k}$. The graph fits well. To reduce magnitude of error, we increase the magnitude of time delay. It is obvious that these curves fit better than the curves in Figure 31. It follows that the interconnected dynamical system with jump process incorporated into it performs better than the one without jump.

Chapter 11
Forecasting

11.1 Introduction

In this chapter, we shall sketch an outline about forecasting problem for the case where there is no jump. The sketch for the case where jump exist is similar. An $\epsilon-$ sub-optimal simulated value ($\mathbf{y}^s(\hat{m}_k^{i-1}, t_k^{i-1})$, $\mathbf{p}^s(\hat{m}_k^{i-1}, t_k^{i-1})$) at time t_k^{i-1}, $i \in I(1, K^*)$, are used to define a forecast ($\mathbf{y}^f(\hat{m}_k^{i-1}, t_k^{i-1})$, $\mathbf{p}^f(\hat{m}_k^{i-1}, t_k^{i-1})$) for ($\mathbf{y}(t_k^{i-1}), \mathbf{p}(t_k^{i-1})$) at the time t_k^{i-1} for the system of energy commodity model.

11.2 Forecasting for Energy Commodity Model

In the context of Illustration 9.6.1, for $i \in I(1, K^*)$, we begin forecasting from time t_k^{i-1} . Using the data set up to time t_{k-1}^{i-1}, we compute $\hat{m}_a^{i-1}$, $\hat{m}_a^{i-1}$, $u_j(\hat{m}_a^{i-1}, t_a^{i-1})$, $\beta_j(\hat{m}_a^{i-1}, t_a^{i-1})$, $\kappa_{j,l}(\hat{m}_a^{i-1}, t_a^{i-1})$, $\gamma_{j,l}(\hat{m}_a^{i-1}, t_a^{i-1})$, $\delta_{j,l}(\hat{m}_a^{i-1}, t_a^{i-1})$, $\sigma_{j,l}(\hat{m}_a^{i-1}, t_a^{i-1})$, $j, l \in I(1, 3)$ for $a \in I(0, k-1)$. We assume that we have no information about the real data set $\{y_j(t_a^{i-1})\}_{a=k}^{N_{i-1}}$. Under these considerations, imitating the computational procedure outlined in Section 10 and using solutions to (9.58)-(9.59), we find the estimate of the forecast $\mathbf{y}^f(\hat{m}_k^{i-1}, t_k^{i-1})$ and $\mathbf{p}^f(\hat{m}_k^{i-1}, t_k^{i-1})$ at time t_k^{i-1} as follows;

$$\left\{\begin{array}{lcl} y_j^f(\hat{m}_k^{i-1}, t_k^{i-1}) & = & y_j^s(\hat{m}_{k-1}^{i-1}, t_{k-1}^{i-1}) + \Big(u_j(\hat{m}_{k-1}^{i-1}, t_{k-1}^{i-1}) - y_j^s(\hat{m}_{k-1}^{i-1}, t_{k-1}^{i-1})\Big) \times \\ & & \Big[\kappa_{j,j}(\hat{m}_{k-1}^{i-1}, t_{k-1}^{i-1}) y_j^s(\hat{m}_{k-1}^{i-1}, t_{k-1}^{i-1}) \\ & & + \sum\limits_{l \neq j}^{n} \kappa_{j,l} y_l^s(\hat{m}_{k-1}^{i-1}, t_{k-1}^{i-1})\Big] \Delta t \\ & & +\delta_{j,j}(\hat{m}_{k-1}^{i-1}, t_{k-1}^{i-1}) \Big(u_j(\hat{m}_{k-1}^{i-1}, t_{k-1}^{i-1}) - y_j^s(\hat{m}_{k-1}^{i-1}, t_{k-1}^{i-1})\Big) W_{j,j}(k) \\ & & + \Big(u_j(\hat{m}_{k-1}^{i-1}, t_{k-1}^{i-1}) - y_j^s(\hat{m}_{k-1}^{i-1}, t_{k-1}^{i-1})\Big) \sum\limits_{l \neq j}^{n} \delta_{j,l} y_l^s(\hat{m}_{k-1}^{i-1}, t_{k-1}^{i-1}) W_{j,l}(k), \end{array}\right. \tag{11.1}$$

$$
\begin{cases}
p_j^f(\hat{m}_k^{i-1}, t_k^{i-1}) & = & p_j^s(\hat{m}_{k-1}^{i-1}, t_{k-1}^{i-1}) + p_j^s(\hat{m}_{k-1}^{i-1}, t_{k-1}^{i-1}) \Big[\gamma_{j,j}(\hat{m}_{k-1}^{i-1}, t_{k-1}^{i-1}) \Big(y_j^s(\hat{m}_{k-1}^{i-1}, t_{k-1}^{i-1}) \\
& & -p_j^s(\hat{m}_{k-1}^{i-1}, t_{k-1}^{i-1})\Big) + \beta_j(\hat{m}_{k-1}^{i-1}, t_{k-1}^{i-1}) \\
& & + \sum_{l \neq j}^{n} \gamma_{j,l}(\hat{m}_{k-1}^{i-1}, t_{k-1}^{i-1}) p_l^s(\hat{m}_{k-1}^{i-1}, t_{k-1}^{i-1}) \Big] \Delta t \\
& & +\sigma_{j,j}(\hat{m}_{k-1}^{i-1}, t_{k-1}^{i-1}) p_j^s(\hat{m}_{k-1}^{i-1}, t_{k-1}^{i-1}) Z_{j,j}(k) \\
& & +p_j^s(\hat{m}_{k-1}^{i-1}, t_{k-1}^{i-1}) \sum_{l \neq j}^{n} \sigma_{j,l}(\hat{m}_{k-1}^{i-1}, t_{k-1}^{i-1}) p_l^s(\hat{m}_{k-1}^{i-1}, t_{k-1}^{i-1}) Z_{j,l}(k),
\end{cases}
\tag{11.2}
$$

where the estimates $u_j(\hat{m}_{k-1}^{i-1}, t_{k-1}^{i-1})$, $\beta_j(\hat{m}_{k-1}^{i-1}, t_{k-1}^{i-1})$, $\kappa_{j,l}(\hat{m}_{k-1}^{i-1}, t_{k-1}^{i-1})$, $\gamma_{j,l}(\hat{m}_{k-1}^{i-1}, t_{k-1}^{i-1})$, $\delta_{j,l}(\hat{m}_{k-1}^{i-1}, t_{k-1}^{i-1})$, $\sigma_{j,l}(\hat{m}_{k-1}^{i-1}, t_{k-1}^{i-1})$, $j, l \in I(1,3)$ are estimated with respect to the known past data set up to the time t_{k-1}^{i-1}. We note that $\mathbf{y}^f_{\hat{m}_k^{i-1}, t_k^{i-1}}$ is the ϵ-sub-optimal estimate for $\mathbf{y}_j(t_k^{i-1})$ at time t_k^{i-1}.

To determine ($\mathbf{y}^f(\hat{m}_{k+1}^{i-1}, t_{k+1}^{i-1}), \mathbf{p}^f(\hat{m}_{k+1}^{i-1}, t_{k+1}^{i-1})$), we need $u_j(\hat{m}_k^{i-1}, t_k^{i-1})$, $\beta_j(\hat{m}_k^{i-1}, t_k^{i-1})$, $\kappa_{j,l}(\hat{m}_k^{i-1}, t_k^{i-1})$, $\gamma_{j,l}(\hat{m}_k^{i-1}, t_k^{i-1})$, $\delta_{j,l}(\hat{m}_k^{i-1}, t_k^{i-1})$, and $\sigma_{j,l}(\hat{m}_k^{i-1}, t_k^{i-1})$, $j, l \in I(1,3)$. Since we only have information of real data up to time t_{k-1}, we use the forecasted estimate $y_j^f(\hat{m}_k^{i-1}, t_k^{i-1})$ as the estimate of $y_j(t_k^{i-1})$ and to estimate $u_j(\hat{m}_k^{i-1}, t_k^{i-1})$, $\beta_j(\hat{m}_k^{i-1}, t_k^{i-1})$, $\kappa_{j,l}(\hat{m}_k^{i-1}, t_k^{i-1})$, $\gamma_{j,l}(\hat{m}_k^{i-1}, t_k^{i-1})$, $\delta_{j,l}(\hat{m}_k^{i-1}, t_k^{i-1})$, and $\sigma_{j,l}(\hat{m}_k^{i-1}, t_k^{i-1})$, $j, l \in I(1,3)$.
Hence, we can write $u_j(\hat{m}_k^{i-1}, t_k^{i-1})$ as

$$
\begin{cases}
u_j(\hat{m}_k^{i-1}, t_k^{i-1}) & \equiv & u_j(\hat{m}_k^{i-1}, y_j(t_{k-\hat{m}_k^{i-1}+1}^{i-1}), y_j(t_{k-\hat{m}_k^{i-1}+2}^{i-1}), ..., y_j(t_{k-1}^{i-1}), y_j^f(\hat{m}_k^{i-1}, t_k^{i-1})) \\
\kappa_{j,l}(\hat{m}_k^{i-1}, t_k^{i-1}) & \equiv & \kappa_{j,l}(\hat{m}_k^{i-1}, y_j(t_{k-\hat{m}_k^{i-1}+1}^{i-1}), y_j(t_{k-\hat{m}_k^{i-1}+2}^{i-1}), ..., y_j(t_{k-1}^{i-1}), y_j^f(\hat{m}_k^{i-1}, t_k^{i-1})) \\
\delta_{j,l}(\hat{m}_k^{i-1}, t_k^{i-1}) & \equiv & \delta_{j,l}(\hat{m}_k^{i-1}, y_j(t_{k-\hat{m}_k^{i-1}+1}^{i-1}), y_j(t_{k-\hat{m}_k^{i-1}+2}^{i-1}), ..., y_j(t_{k-1}^{i-1}), y_j^f(\hat{m}_k^{i-1}, t_k^{i-1})) \\
\beta_{j,l}(\hat{m}_k^{i-1}, t_k^{i-1}) & \equiv & \beta_{j,l}(\hat{m}_k^{i-1}, p_j(t_{k-\hat{m}_k^{i-1}+1}^{i-1}), p_j(t_{k-\hat{m}_k^{i-1}+2}^{i-1}), ..., p_j(t_{k-1}^{i-1}), p_j^f(\hat{m}_k^{i-1}, t_k^{i-1})) \\
\gamma_{j,l}(\hat{m}_k^{i-1}, t_k^{i-1}) & \equiv & \gamma_{j,l}(\hat{m}_k^{i-1}, p_j(t_{k-\hat{m}_k^{i-1}+1}^{i-1}), p_j(t_{k-\hat{m}_k^{i-1}+2}^{i-1}), ..., p_j(t_{k-1}^{i-1}), p_j^f(\hat{m}_k^{i-1}, t_k^{i-1})) \\
\sigma_{j,l}(\hat{m}_k^{i-1}, t_k^{i-1}) & \equiv & \sigma_{j,l}(\hat{m}_k^{i-1}, p_j(t_{k-\hat{m}_k^{i-1}+1}^{i-1}), p_j(t_{k-\hat{m}_k^{i-1}+2}^{i-1}), ..., p_j(t_{,k-1}^{i-1}), p_j^f(\hat{m}_k^{i-1}, t_k^{i-1})), \\
& & j, l \in I(1,n).
\end{cases}
$$

To find ($y_j^f(\hat{m}_{k+2}^{i-1}, t_{k+2}^{i-1}), p_j^f(\hat{m}_{k+2}^{i-1}, t_{k+2}^{i-1})$), we use the estimates

$$
\begin{cases}
u_j(\hat{m}_{k+1}^{i-1}, t_{k+1}^{i-1}) & \equiv & u_j(\hat{m}_{k+1}^{i-1}, y_j(t_{k-\hat{m}_k^{i-1}+2}^{i-1}), y_j(t_{k-\hat{m}_k^{i-1}+3}^{i-1}), ..., y_j(t_{k-1}^{i-1}), \\
& & y_j^f(\hat{m}_k^{i-1}, t_k^{i-1}), y_j^f(\hat{m}_{k+1,y_1}^{i-1}, k+1)) \\
\kappa_{j,l}(\hat{m}_{k+1}^{i-1}, t_{k+1}^{i-1}) & \equiv & \kappa_{j,l}(\hat{m}_{k+1}^{i-1}, y_j(t_{k-\hat{m}_k^{i-1}+2}^{i-1}), y_j(t_{k-\hat{m}_k^{i-1}+3}^{i-1}), ..., y_j(t_{k-1}^{i-1}), \\
& & y_j^f(\hat{m}_k^{i-1}, t_k^{i-1}), y_j^f(\hat{m}_{k+1}^{i-1}, t_{k+1}^{i-1}))
\end{cases}
$$

$$\begin{cases} \delta_{j,l}(\hat{m}^{i-1}_{k+1}, t^{i-1}_{k+1}) & \equiv & \delta_{j,l}(\hat{m}^{i-1}_{k+1}, y_j(t^{i-1}_{k-\hat{m}^{i-1}_k+2}), y_j(t^{i-1}_{k-\hat{m}^{i-1}_k+3}), ..., y_j(t^{i-1}_{k-1}), \\ & & y^f_j(\hat{m}^{i-1}_k, t^{i-1}_k), y^f_j(\hat{m}^{i-1}_{k+1}, t^{i-1}_{k+1})) \\ \beta_{j,l}(\hat{m}^{i-1}_{k+1}, t^{i-1}_{k+1}) & \equiv & \beta_{j,l}(\hat{m}^{i-1}_{k+1}, p_j(t^{i-1}_{k-\hat{m}^{i-1}_k+2}), p_j(t^{i-1}_{k-\hat{m}^{i-1}_k+3}), ..., p_j(t^{i-1}_{k-1}), \\ & & p^f_j(\hat{m}^{i-1}_k, t^{i-1}_k), p^f_j(\hat{m}^{i-1}_{k+1}, t^{i-1}_{k+1})) \\ \gamma_{j,l}(\hat{m}^{i-1}_{k+1}, t^{i-1}_{k+1}) & \equiv & \gamma_{j,l}(\hat{m}^{i-1}_{k+1}, p_j(t^{i-1}_{k-\hat{m}^{i-1}_k+2}), p_j(t^{i-1}_{k-\hat{m}^{i-1}_k+3}), ..., p_j(t^{i-1}_{k-1}), \\ & & p^f_j(\hat{m}^{i-1}_k, t^{i-1}_k), p^f_j(\hat{m}^{i-1}_{k+1}, t^{i-1}_{k+1})) \\ \sigma_{j,l}(\hat{m}^{i-1}_{k+1}, t^{i-1}_{k+1}) & \equiv & \sigma_{j,l}(\hat{m}^{i-1}_{k+1}, p_j(t^{i-1}_{k-\hat{m}^{i-1}_k+2}), p_j(t^{i-1}_{k-\hat{m}^{i-1}_k+3}), ..., p_j(t^{i-1}_{k-1}), \\ & & p^f_j(\hat{m}^{i-1}_k, t^{i-1}_k), p^f_j(\hat{m}^{i-1}_{k+1}, t^{i-1}_{k+1})),\ j, l \in I(1, n). \end{cases}$$

Continuing this process in this manner, to find $\left(y^f_j(\hat{m}^{i-1}_{k+l}, t^{i-1}_{k+l}), p^f_j(\hat{m}^{i-1}_{k+l}, t^{i-1}_{k+l})\right)$, we use the estimates

$$\begin{cases} u_j(\hat{m}^{i-1}_{k+l-1}, t^{i-1}_{k+l-1}) & \equiv & u_j(\hat{m}^{i-1}_{k+l-1}, y_j(t^{i-1}_{k-\hat{m}^{i-1}_k+l}), y_j(t^{i-1}_{k-\hat{m}^{i-1}_k+l+1}), ..., y_j(t^{i-1}_{k-1}), \\ & & y^f_j(\hat{m}^{i-1}_k, t^{i-1}_k), ..., y^f_j(\hat{m}^{i-1}_{k+l-1}, t^{i-1}_{k+l-1})) \\ \kappa_{j,l}(\hat{m}^{i-1}_{k+l-1}, t^{i-1}_{k+l-1}) & \equiv & \kappa_{j,l}(\hat{m}^{i-1}_{k+l-1}, y_j(t^{i-1}_{k-\hat{m}^{i-1}_k+l}), y_j(t^{i-1}_{k-\hat{m}^{i-1}_k+l+1}), ..., y_j(t^{i-1}_{k-1}), \\ & & y^f_j(\hat{m}^{i-1}_k, t^{i-1}_k), ..., y^f_j(\hat{m}^{i-1}_{k+l-1}, t^{i-1}_{k+l-1})) \\ \delta_{j,l}(\hat{m}^{i-1}_{k+l-1}, t^{i-1}_{k+l-1}) & \equiv & \delta_{j,l}(\hat{m}^{i-1}_{k+l-1}, y_j(t^{i-1}_{k-\hat{m}^{i-1}_k+l}), y_j(t^{i-1}_{k-\hat{m}^{i-1}_k+l+1}), ..., y_j(t^{i-1}_{k-1}), \\ & & y^f_j(\hat{m}^{i-1}_k, t^{i-1}_k), ..., y^f_j(\hat{m}^{i-1}_{k+l-1}, t^{i-1}_{k+l-1})) \\ \beta_{j,l}(\hat{m}^{i-1}_{k+l-1}, t^{i-1}_{k+l-1}) & \equiv & \beta_{j,l}(\hat{m}^{i-1}_{k+l-1}, p_j(t^{i-1}_{k-\hat{m}^{i-1}_k+l}), p_j(t^{i-1}_{k-\hat{m}^{i-1}_k+l+1}), ..., p_j(t^{i-1}_{k-1}), \\ & & p^f_j(\hat{m}^{i-1}_k, t^{i-1}_k), ..., p^f_j(\hat{m}^{i-1}_{k+l-1}, t^{i-1}_{k+l-1})) \\ \gamma_{j,l}(\hat{m}^{i-1}_{k+l-1}, t^{i-1}_{k+l-1}) & \equiv & \gamma_{j,l}(\hat{m}^{i-1}_{k+l-1}, p_j(t^{i-1}_{k-\hat{m}^{i-1}_k+l}), p_j(t^{i-1}_{k-\hat{m}^{i-1}_k+l+1}), ..., p_j(t^{i-1}_{k-1}), \\ & & p^f_j(\hat{m}^{i-1}_k, t^{i-1}_k), ..., p^f_j(\hat{m}^{i-1}_{k+l-1}, t^{i-1}_{k+l-1})) \\ \sigma_{j,l}(\hat{m}^{i-1}_{k+l-1}, t^{i-1}_{k+l-1}) & \equiv & \sigma_{j,l}(\hat{m}^{i-1}_{k+l-1}, p_j(t^{i-1}_{k-\hat{m}^{i-1}_k+i}), p_j(t^{i-1}_{k-\hat{m}^{i-1}_k+l+1}), ..., p_j(t^{i-1}_{k-1}), \\ & & p^f_j(\hat{m}^{i-1}_k, t^{i-1}_k), ..., p^f_j(\hat{m}^{i-1}_{k+l-1}, t^{i-1}_{k+l-1})),\ j, l \in I(1, n). \end{cases}$$

11.2.1 Prediction/Confidence Interval for Energy Commodities

In order to be able to assess the future certainty, we also discuss about the prediction/confidence interval. We define the $100(1-\alpha)\%$ confidence interval for the forecast of the state $(\mathbf{y}^f_{\hat{m}^{i-1}_l, t^{i-1}_l}, \mathbf{p}^f_{\hat{m}^{i-1}_l, t^{i-1}_l})$ at time t^{i-1}_l, $l \geq k$, $j \in I(1, n)$ as

$$\begin{cases} y^f_j(\hat{m}^{i-1}_l, t^{i-1}_l) \pm z_{1-\alpha/2}\left(s^{j,j}_{\hat{m}^{i-1}_{l-1}, t^{i-1}_{l-1}}(y^f_j)\right)^{1/2}, \\ p^f_j(\hat{m}^{i-1}_l, t^{i-1}_l) \pm z_{1-\alpha/2}\left(s^{j,j}_{\hat{m}^{i-1}_{l-1}, t^{i-1}_{l-1}}(p^f_j)\right)^{1/2}, \end{cases}$$

where $\left(s^{j,j}(\hat{m}_{l-1}^{i-1}, t_{l-1}^{i-1})\right)^{1/2}$ is the estimate for the sample standard deviation for the forecasted state y_j, and $\left(s^{j,j}(\hat{m}_{l-1}^{i-1}, t_{l-1}^{i-1})\right)^{1/2}$ is the estimate for the sample standard deviation for the forecasted state p_j derived from the following iterative process

$$\begin{cases} y_j^f(\hat{m}_l^{i-1}, t_l^{i-1}) &= y_j^f(\hat{m}_{l-1}^{i-1}, t_{l-1}^{i-1}) + \left(u_j(\hat{m}_{l-1}^{i-1}, t_{l-1}^{i-1}) - y_j^f(\hat{m}_{l-1}^{i-1}, t_{l-1}^{i-1})\right) \times \\ & \left[\kappa_{j,j}(\hat{m}_{l-1}^{i-1}, t_{l-1}^{i-1}) y_j^f(\hat{m}_{l-1}^{i-1}, t_{l-1}^{i-1}) + \sum_{l \neq j}^{n} \kappa_{j,l} y_l^f(\hat{m}_{l-1}^{i-1}, t_{l-1}^{i-1})\right] \Delta t \\ & +\delta_{j,j}(\hat{m}_{l-1}^{i-1}, t_{l-1}^{i-1}) \left(u_j(\hat{m}_{l-1}^{i-1}, t_{l-1}^{i-1}) - y_j^f(\hat{m}_{l-1}^{i-1}, t_{l-1}^{i-1})\right) W_{j,j}(l) \\ & + \left(u_j(\hat{m}_{l-1}^{i-1}, t_{l-1}^{i-1}) - y_j^f(\hat{m}_{l-1}^{i-1}, t_{l-1}^{i-1})\right) \sum_{l \neq j}^{n} \delta_{j,l} y_l^f(\hat{m}_{l-1}^{i-1}, t_{l-1}^{i-1}) W_{j,l}(l), \\ p_j^f(\hat{m}_l^{i-1}, t_l^{i-1}) &= p_j^f(\hat{m}_{l-1}^{i-1}, t_{l-1}^{i-1}) + p_j^f(\hat{m}_{l-1}^{i-1}, t_{l-1}^{i-1}) \Big[\gamma_{j,j}(\hat{m}_{l-1}^{i-1}, t_{l-1}^{i-1}) \Big(y_j^f(\hat{m}_{l-1}^{i-1}, t_{l-1}^{i-1}) \\ & -p_j^f(\hat{m}_{l-1}^{i-1}, t_{l-1}^{i-1})\Big) + \beta_j(\hat{m}_{l-1}^{i-1}, t_{l-1}^{i-1}) \\ & + \sum_{l \neq j}^{n} \gamma_{j,l}(\hat{m}_{l-1}^{i-1}, t_{l-1}^{i-1}) p_l^f(\hat{m}_{l-1}^{i-1}, t_{l-1}^{i-1})\Big] \Delta t \\ & +\sigma_{j,j}(\hat{m}_{l-1}^{i-1}, t_{l-1}^{i-1}) p_j^f(\hat{m}_{l-1}^{i-1}, t_{l-1}^{i-1}) Z_{j,j}(l) \\ & +p_j^f(\hat{m}_{l-1}^{i-1}, t_{l-1}^{i-1}) \sum_{l \neq j}^{n} \sigma_{j,l}(\hat{m}_{l-1}^{i-1}, t_{l-1}^{i-1}) p_l^f(\hat{m}_{l-1}^{i-1}, t_{l-1}^{i-1}) Z_{j,l}(l). \end{cases} \tag{11.3}$$

It is clear that the 95 $\%$ confidence interval for the forecast at time t_k is

$$\begin{cases} \left(y_j^f(\hat{m}_k^{i-1}, t_k^{i-1}) - 1.96\left(s^{j,j}_{\hat{m}_{k-1}^{i-1},k-1}(y_j^f)\right)^{1/2}, y_j^f(\hat{m}_k^{i-1}, t_k^{i-1}) + 1.96\left(s^{j,j}_{\hat{m}_{k-1}^{i-1},k-1}(y_j^f)\right)^{1/2}\right), \\ \left(p_j^f(\hat{m}_k^{i-1}, t_k^{i-1}) - 1.96\left(s^{j,j}_{\hat{m}_{k-1}^{i-1},k-1}(p_j^f)\right)^{1/2}, p_j^f(\hat{m}_k^{i-1}, t_k^{i-1}) + 1.96\left(s^{j,j}_{\hat{m}_{k-1}^{i-1},k-1}(p_j^f)\right)^{1/2}\right), \end{cases}$$

where the lower ends denote the lower bounds of the state estimate and the upper ends denote the upper bounds of the state estimate.

11.2.2 Illustration: Prediction/Confidence Interval for Energy Commodities with no jump

For the case of no jump, the following graphs show the simulated, forecasted and 95 percent confidence limit for the daily Henry Hub Natural gas data set [27], daily Crude Oil data set [28], and daily Coal data set [26], respectively.

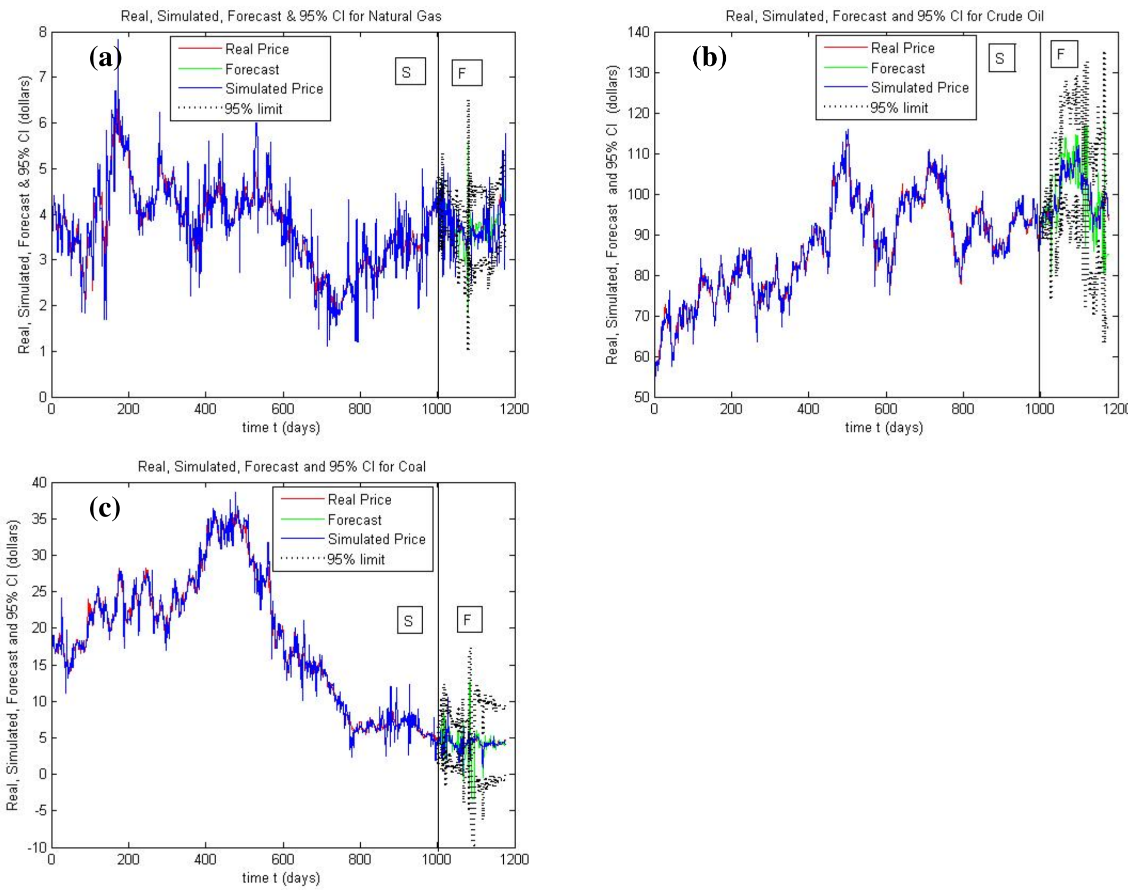


Figure 39.: Real, Simulated, Forecasted Prices and 95% C.I. with no jump.

Figures 39: **(a), (b), (c)** show the graph of the forecast and 95 percent confidence limit for the case where there is no jump for the daily Henry Hub Natural gas data set [27], daily Crude Oil data set [28], and daily Coal data set [26], respectively. Figures 39: **(a)**, **(b)**, and **(c)** show two region: the simulation region S and the forecast region F. For the simulation region S, we plot the real data set together with the simulated data set as described in Figure 38. For the forecast region F, we plot the estimate of the forecast as explained in Section 11. The upper and the lower simulated sketches in Figure 39 **(a), (b), and (c)** are corresponding to the upper and lower ends of the 95% confidence interval.

11.2.3 Illustration: Prediction/Confidence Interval for Energy Commodities with Jump

For the case of jump process, the following graphs show the forecast and 95 percent confidence limit for the daily Henry Hub Natural gas data set [27], daily Crude Oil data set [28], and daily Coal data set [26], respectively.

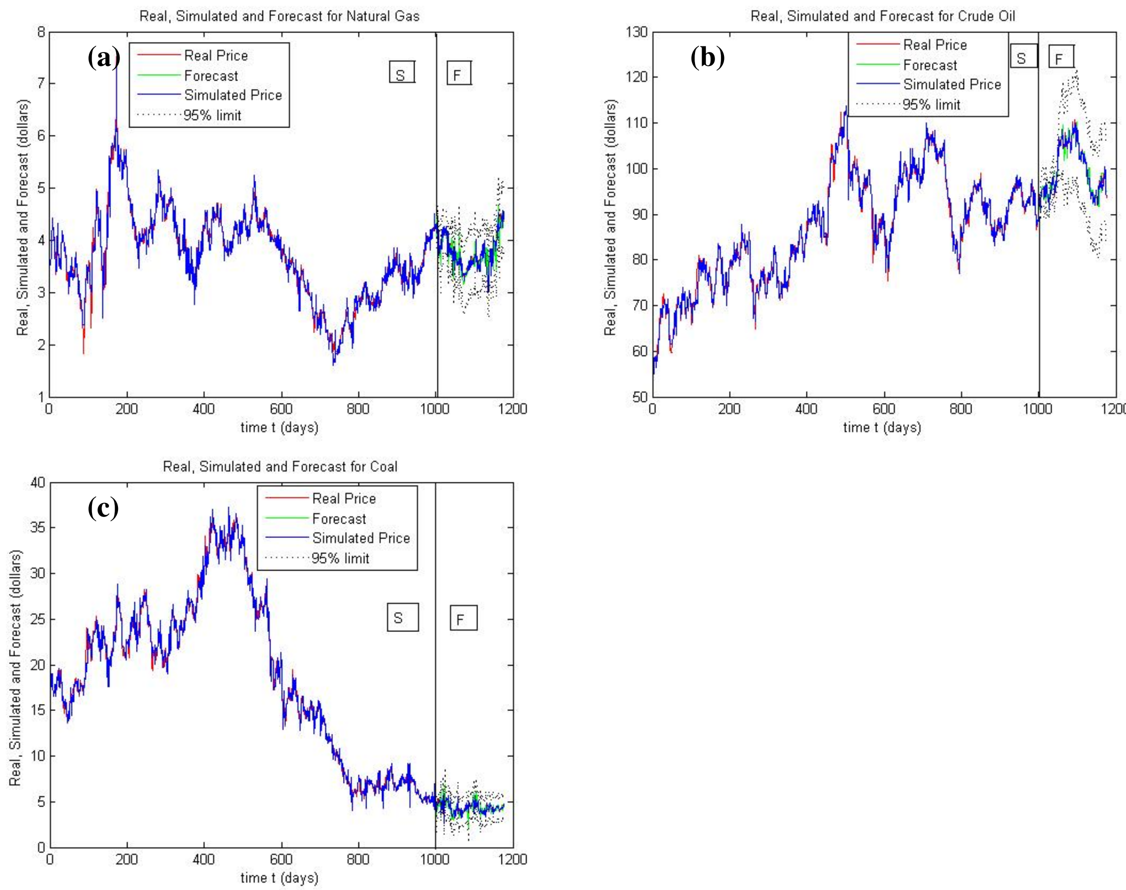


Figure 40.: Real, Simulated, Forecasted Prices and 95% CI with jump.

Figures 40: **(a), (b), (c)** show the graph of the forecast and 95 percent confidence limit for the case where there is jump for the daily Henry Hub Natural gas data set [27], daily Crude Oil data set [28], and daily Coal data set [26], respectively. Figure 40: **(a)**, **(b)**, and **(c)** show two region: the simulation region S and the forecast region F. For the simulation region S, we plot the real data set together with the simulated data set as described in Figure 38.

Chapter 12

Conclusion and Future Work

It is easily seen from Figures 31 and 38 that the model with jump performs better than the model without jump. This is because the curve fits better at the jump times for the jump case than the case without jump.

For $j, l \in I(1,3)$, the estimates for the drift interaction parameters $\gamma_{j,l}(\hat{m}_k, t_k^{i-1})$ for the case where jump is incorporated suggest that there is definitely interactions between the spot price of these three commodities. As discussed in (9.46) and (9.47), the sign of these parameters suggest if there is competition or cooperation between commodity l and j. The estimate of the parameter $\gamma_{j,l}(\hat{m}_k, t_k^{i-1})$ in Tables 17 and 24 and Figures 28 and 35 suggest that these commodities either compete or cooperate with each other depending on the time period. We can also describe the relationship between any two commodity j and l, $j \neq l \in I(1,3)$ based on the overall average $\bar{\gamma}_{j,l} = \frac{1}{N}\sum_{k=1}^{N} \gamma_{j,l}(\hat{m}_k, t_k^{i-1})$. For example, for the case where jump is incorporated, $\bar{\gamma}_{1,3} = -0.0017$ and $\bar{\gamma}_{3,1} = -0.0095$. This suggests that on the average, there is competition between these two commodities. Also, $\bar{\gamma}_{1,2} = 0.0018$ and $\bar{\gamma}_{2,1} = 0.0083$. This indicates that on the average, there is cooperation between natural gas and crude oil. Finally, $\bar{\gamma}_{2,3} = -0.0146$ and $\bar{\gamma}_{3,2} = -0.0013$. Therefore, on the average, there is competition between crude oil and coal.

In the future, we plan to apply the Local Lagged Adapted Generalized Method of Moments to interconnected nonlinear stochastic dynamic model for log-spot price, expected log-spot price and volatility process. Also, we plan to incorporate delay in the multivariate interconnected nonlinear stochastic model. We plan to be able to apply the extended Local Lagged Adapted Generalized Method of Moments to other multivariate interconnected nonlinear dynamic model different from energy commodity model.

Appendix A

A.1 Existence and Positivity of delayed Volatility in Chapter 4

LEMMA A.1 *Suppose $u(t)$ is $\mathcal{F}_t$ square-integrable, adapted, non-anticipative process. We have*

$$\mathbb{E}\left[\left(\int_{t_0}^{\infty} u(t)dW(t)\right)^4\right] = 0. \tag{A.1}$$

Proof. We start by showing that $(A.1)$ holds for simple predictable process using Definition 1.4.2. The extension of the stochastic integral to square-integrable adapted process follows from [91].

We denote $W(t_i)$ by W_{t_i}. Using (1.7), we have

$$\begin{aligned}
\mathbb{E}\left[\left(\int_{t_0}^{\infty} u(t)dW(t)\right)^4\right] &= \mathbb{E}\left[\left(\sum_{i=1}^{n} F_i(W_{t_i} - W_{t_{i-1}})\right)^4\right] \\
&= 3\sum_{i=1}^{n} \mathbb{E}[F_i^4]\Delta t_i^2 + \sum_{\substack{i,j=1 \\ i\neq j}}^{n} \mathbb{E}\left[F_i^2\right]\left[F_j^2\right]\Delta t_i \Delta t_j \\
&= 0
\end{aligned}$$

□

Denote by $C = C([-\tau, 0], \mathbb{R})$ the Banach space of all continuous functions $[-\tau, 0] \to \mathbb{R}$ under the supremum norm

$$||\alpha||_C = \sup_{\theta\in[-\tau,0]} |\alpha(\theta)|, \ \ \alpha \in C. \tag{A.2}$$

Denote $L^2(C, \mathbb{R})$ to be the Banach space of all measurable maps $[t_0, T] \times \Omega \to \mathbb{R}$ which are L^2 with norm

$$||\beta||^2_{L^2(C,\mathbb{R})} = \mathbb{E}_\Omega[|\beta|^2_C]. \tag{A.3}$$

Define $u(t, \psi_t) \equiv \sigma^2(t, \psi_t)$. The differential equation (4.10) reduces to

$$du(t,\psi_t) = f(t,u(t,\psi_t),w)dt, \;\; u(t_0,\psi) = u_0(\psi) > 0. \tag{A.4}$$

where

$$f(t,\psi_t,w) = \alpha + \beta\left[\int_{t-\tau}^{t}\sqrt{u(s,\psi_s)}dW_2(s)\right]^2 + cu(t,\psi_t). \tag{A.5}$$

THEOREM A.1 *Using Lemma A.1, $f(t,\psi_t,w)$ defined in (A.5) satisfies*

$$\begin{cases} ||f(t,u_1,w) - f(t,u_2,w)||_{L^2([t_0,T]\times L^2(\Omega,\mathbb{R})\times\Omega,\mathbb{R})} & \leq \;\; L||u_1-u_2||_{L^2([t_0,T]\times C,\mathbb{R})} \\ ||f(t,u,w)|| & \leq \;\; K(1+||u||_{L^2([t_0,T]\times C,\mathbb{R})}) \end{cases} \tag{A.6}$$

for all u_1, $u_2 \in L^2([t_0,T]\times C,\mathbb{R})$

Proof. For any $u_1, u_2 \in L^2([t_0,T]\times C,\mathbb{R})$

$$\begin{aligned}
&\|f(t,u_1,w) - f(t,u_2,w)\|_{L^2}^2 = \\
&\mathbb{E}\left|\beta\left[\left(\int_{t-\tau}^{t}\sqrt{u_1(s,\psi_s)}dW(s)\right)^2 - \left(\int_{t-\tau}^{t}\sqrt{u_2(s,\psi_s)}dW(s)\right)^2\right]\right. \\
&\left.+c(u_1(t,\psi_t)-u_2(t,\psi_t))\right|^2 \\
&\leq 2\mathbb{E}\left[\beta^2\left|\left[\left(\int_{t-\tau}^{t}\sqrt{u_1(s,\psi_s)}dW(s)\right)^2 - \left(\int_{t-\tau}^{t}\sqrt{u_2(s,\psi_s)}dW(s)\right)^2\right]\right|^2\right. \\
&\left.+c^2\left|u_1(t,\psi_t)-u_2(t,\psi_t)\right|^2\right] \\
&\leq 4\beta^2\mathbb{E}\left[\left|\left(\int_{t-\tau}^{t}\sqrt{u_1(s,\psi_s)}dW(s)\right)\right|^4\right] + 4\beta^2\mathbb{E}\left[\left|\left(\int_{t-\tau}^{t}\sqrt{u_2(s,\psi_s)}dW(s)\right)\right|^4\right] \\
&+2c^2\mathbb{E}\left|u_1(t,\psi_t)-u_2(t,\psi_t)\right|^2 \\
&\leq L\|u_1(t,\psi_t)-u_2(t,\psi_t)\|_{L^2}^2, \;\; \text{where} \;\; L = 2c^2.
\end{aligned}$$

Likewise,

$$\begin{aligned}
\|f(t,u,w)\|_{L^2}^2 \;\; &= \;\; \mathbb{E}\left|\alpha + \beta\left(\int_{t-\tau}^{t}\sqrt{u(s,\psi_s)}dW(s)\right)^2 + cu(t,\psi_t)\right|^2 \\
&\leq \;\; 2b^2\mathbb{E}\left|\left(\int_{t-\tau}^{t}\sqrt{u(s,\psi_s)}dW(s)\right)\right|^4 + 2\mathbb{E}\left|\alpha + cu(t,\psi_t)\right|^2 \\
&\leq \;\; 4\mathbb{E}\left[|\alpha|^2 + c^2\left|u(t,\psi_t)\right|^2\right] \\
&\leq \;\; K\left[1+\|u\|_{L_2}\right], \quad \text{where} \;\; K = 4\max\{|\alpha|^2, |c|^2\}.
\end{aligned}$$

$\square$

Next, we shall show that the solution $u(t,\psi)$ of the IVP $(A.4)$ satisfies Lipschitz condition whenever the initial condition $y(t_0,\psi) = y_0(\psi)$ in $(A.4)$ satisfies the following assumption:

H_2:

a). $u_0(\psi)$ satisfies Lipschitz condition, that is, for every $\psi_1,\ \psi_2 \in C$, there exist a constant $M_1 > 0$, such that

$$\|u_0(\psi_1) - u_0(\psi_2)\| \le M_1\|\psi_1 - \psi_2\|,$$

and

b). $\inf\limits_{\psi} u_0(\psi) = M_2 > 0.$

THEOREM A.2 *Every solution $u(t,\psi_t)$ satisfying $(A.4)$ with initial condition satisfying H_2(a) satisfies Lipschitz condition.*

Furthermore, under condition H_2(b), then $\sqrt{u(t,\psi_t)}$ satisfies Lipschitz condition.

Proof. For any solution $u_1(t,\psi_1)$, $u_2(t,\psi_2)$ satisfying $(A.4)$ and assumptions H_2, with $\psi_i \equiv \psi_{it}$, $i = 1,2$, we have

$$\begin{aligned}
&\|u_1(t,\psi_1) - u_2(t,\psi_2)\|^2_{L^2} = \\
&\mathbb{E}\left|\int_{t_0}^{t}\left(\beta\left[\left(\int_{s-\tau}^{s}\sqrt{u_1(r,\psi_1)}dW(r)\right)^2 - \left(\int_{s-\tau}^{s}\sqrt{u_2(r,\psi_2)}dW(r)\right)^2\right]\right.\right. \\
&\left.\left. +c(u_1(s,\psi_1) - u_2(s,\psi_2))\right) ds\right|^2 \\
&\le \mathbb{E}\left|\int_{t_0}^{t}\left(\beta\left[\left(\int_{s-\tau}^{s}\sqrt{u_1(r,\psi_1)}dW(r)\right)^2 - \left(\int_{s-\tau}^{s}\sqrt{u_2(r,\psi_2)}dW(r)\right)^2\right]\right) ds\right. \\
&\left. +c\int_{t_0}^{t}(u_1(s,\psi_1) - u_2(s,\psi_2))ds\right|^2 \\
&\le 2\mathbb{E}\left|\int_{t_0}^{t}\left(\beta\left[\left(\int_{s-\tau}^{s}\sqrt{u_1(r,\psi_1)}dW(r)\right)^2 - \left(\int_{s-\tau}^{s}\sqrt{u_2(r,\psi_2)}dW(r)\right)^2\right]\right) ds\right|^2 \\
&+2\mathbb{E}\left|c\int_{t_0}^{t}(u_1(s,\psi_1) - u_2(s,\psi_2))ds\right|^2 \\
&\le 2\mathbb{E}\int_{t_0}^{t}\left|\beta\left[\left(\int_{s-\tau}^{s}\sqrt{u_1(r,\psi_1)}dW(r)\right)^2 - \left(\int_{s-\tau}^{s}\sqrt{u_2(r,\psi_2)}dW(r)\right)^2\right]\right|^2 ds
\end{aligned}$$

$$+2\mathbb{E}\int_{t_0}^{t}|c(u_1(s,\psi_1)-u_2(s,\psi_2))|^2\,ds$$
$$\leq 2b^2T\int_{t_0}^{t}\mathbb{E}\left|\left[\left(\int_{s-\tau}^{s}\sqrt{u_1(r,\psi_1)}dW(r)\right)^2-\left(\int_{s-\tau}^{s}\sqrt{u_2(r,\psi_2)}dW(r)\right)^2\right]\right|^2 ds$$
$$+2c^2T\int_{t_0}^{t}\mathbb{E}\,|(u_1(s,\psi_1)-u_2(s,\psi_2))|^2\,ds,\ \text{ using Holder's inequality,}$$
$$\leq 2c^2T\int_{t_0}^{t}\|(u_1(s,\psi_1)-u_2(s,\psi_2))\|^2 ds,\ \text{ using Lemma } A.1.$$

By Gronwall's inequality, [70, 66], we have

$$\begin{aligned}\|u_1(t,\psi_1)-u_2(t,\psi_2)\| &\leq \|u_1(t_0,\psi_1)-u_2(t_0,\psi_2)\|e^{c^2T^2}\\ &\leq M_3\|u_0(\psi_1)-u_0(\psi_2)\|,\ \ M_3=e^{c^2T^2},\\ &\leq M\|\psi_1-\psi_2\|_0,\ \ \text{where assumption } H_2\text{(a) is used, and } M=M_1M_3.\end{aligned} \tag{A.7}$$

Furthermore, using assumption H_2, there exist a positive constant M_4 such that $M_4 \leq \|\sqrt{u_1(t,\psi_1)}+\sqrt{u_2(t,\psi_2)}\|$. Substituting this into equation $(A.7)$, we have

$$\|\sqrt{u_1(t,\psi_1)}-\sqrt{u_2(t,\psi_2)}\|_{L^2([t_0,T]\times C,\mathbb{R})}\leq N\|\psi_1-\psi_2\|_0,\ \text{ where }\ N=M/M_4. \tag{A.8}$$

□

In the next theorem, we show conditions for positivity of the solution $u(t,\psi_t)$ of $(A.4)$.

THEOREM A.3 *Differential equation in* $(A.4)$ *has a positive solution if* $\alpha>0$, $\beta>0$.

Proof. Using the transformation

$$z(t,\psi_t)=e^{-c(t-t_0)}u(t,\psi_t),$$

equation $(A.4)$ reduces to

$$dz=\left[\alpha e^{-c(t-t_0)}+\beta\left(\int_{t-\tau}^{t}\sqrt{z(s,\psi_s)}exp\left[-\frac{c}{2}(t-s)\right]dW(s)\right)^2\right]dt.$$

Hence, by hypothesis, $z(t,\psi_t)$ is an increasing function of t. Since $u_0(t_0,\psi_t)>0$, then $z(t,\psi_t)$ is positive. □

Appendix B

B.1 Algorithm for Simulation

Algorithm 1 Estimating parameters

Given initial parameters and initial predictions $\hat{x}(t_1|t_0)$ and $P(t_1|t_0)$,

for $k = 1$ to n **do**,

  for $j = 0$ to 2 **do**,

    for $m = 1$ to 6 **do**,

      Compute $\hat{y}(t_k|t_{k-1})$ and $r_{j,m}(t_k|t_{k-1})$

      Compute $\hat{x}(t_k|t_k)$ and $P(t_k|t_k)$ using (5.17),

      Compute $\hat{x}(t_{k+1}|t_k)$ and $P(t_{k+1}|t_k)$ using (5.26),

      Compute e_k using (5.27),

    end for

  end for

end for

Return e_k.

Compute L($\mathbf{\Theta}$) using (5.28),

$\breve{\theta} = \arg\min L(\mathbf{\Theta})$

Return L.

B.2 Expressions in Lemma 5.1

$$
\begin{aligned}
D(t_k|t_{k-1})_{\{\tilde{\mathbf{h}},\tilde{\mathbf{h}}^T\}} &= \frac{\sigma_4-\sigma_2^2}{2h^3}\sum_{p=1}^{n}\left[e_p\mu_p\delta_p\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T + e_p\delta_p^2\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\right] \\
&+\frac{\sigma_2^2}{2h^3}\sum_{\substack{p,q=1\\p\neq q}}^{n}\left[e_q\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T + e_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\right] \\
&+\frac{\sigma_2^2}{4h^4}\sum_{\substack{p,q=1\\p\neq q}}^{n}\left[e_q\delta_q^2\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T + e_p\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T\right] \\
&+\frac{\sigma_2^2}{2h^3}\sum_{\substack{p,q=1\\p\neq q}}^{n}\left[e_q\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T + e_p\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\right] \\
E(t_k|t_{k-1}) &= \frac{\sigma_4-\sigma_2^2}{2h^3}\sum_{p=1}^{n}\left[e_pC^T\mu_p\delta_p\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T + e_pC^T\delta_p^2\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\right] \\
&+\frac{\sigma_2^2}{2h^3}\sum_{\substack{p,q=1\\p\neq q}}^{n}\left[e_qC^T\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T + e_pC^T\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\right] \\
&+\frac{\sigma_2^2}{4h^4}\sum_{\substack{p,q=1\\p\neq q}}^{n}\left[e_qC^T\delta_q^2\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T + e_pC^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T\right] \\
&+\frac{\sigma_2^2}{2h^3}\sum_{\substack{p,q=1\\p\neq q}}^{n}e_qC^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T \\
&+\frac{\sigma_2^2}{2h^3}\sum_{\substack{p,q=1\\p\neq q}}^{n}e_pC^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T
\end{aligned}
$$

$$
\begin{aligned}
J(t_k|t_{k-1}) &= \frac{\sigma_4}{h^3}\sum_{p=1}^{n}e_p\mu_p\delta_p\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T \\
&+ \frac{\sigma_2^2}{h^3}\sum_{\substack{p,q=1\\p\neq q}}^{n}\Big[e_q\mu_p\delta_p\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T + e_q\mu_p\delta_p\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T \\
&\quad +e_p\mu_p\delta_p\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\Big] \\
&+ \frac{\sigma_2^3}{4h^5}\sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n}\Big[e_r\mu_p\delta_p\tilde{\mathbf{h}}^T\delta_q^2\tilde{\mathbf{h}}\mu_p\delta_p\mu_r\delta_r\tilde{\mathbf{h}}^T + e_r\mu_p\delta_p\tilde{\mathbf{h}}^T\delta_q^2\tilde{\mathbf{h}}\mu_r\delta_r\mu_p\delta_p\tilde{\mathbf{h}}^T \\
&\quad +e_q\mu_p\delta_p\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\delta_r^2\tilde{\mathbf{h}}^T\Big]
\end{aligned}
$$

$$
\begin{aligned}
+ \quad & \frac{\sigma_2\sigma_4}{4h^5}\sum_{\substack{p,q\\p\neq q}}^{n}\Big[e_q\mu_p\delta_p\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T + e_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T \\
& + e_q\delta_p^2\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\Big] \\
+ \quad & \frac{\sigma_2^3}{4h^5}\sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n}\Big[e_r\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_r\delta_r\tilde{\mathbf{h}}^T + e_r\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_r\delta_r\mu_p\delta_p\tilde{\mathbf{h}}^T \\
& + e_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\delta_r^2\tilde{\mathbf{h}}^T + e_r\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_r\delta_r\tilde{\mathbf{h}}^T\Big] \\
+ \quad & \frac{\sigma_2\sigma_4}{4h^5}\sum_{\substack{p,q\\p\neq q}}^{n}\Big[e_p\delta_p^2\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}^T + e_q\delta_p^2\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T \\
& + e_q\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\Big] \\
+ \quad & \frac{\sigma_2^3}{4h^5}\sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n}\Big[e_q\delta_p^2\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\delta_r^2\tilde{\mathbf{h}}^T + e_r\delta_p^2\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\mu_r\delta_r\mu_q\delta_q\tilde{\mathbf{h}}^T \\
& + e_r\delta_p^2\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}^T + e_r\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\delta_r\mu_r\mu_q\delta_q\tilde{\mathbf{h}}^T\Big] \\
+ \quad & \frac{\sigma_2^3}{4h^5}\sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n}\Big[e_q\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\delta_r^2\tilde{\mathbf{h}}^T + e_r\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\mu_q\delta_q\delta_r\mu_r\tilde{\mathbf{h}}^T\Big] \\
+ \quad & \frac{\sigma_2\sigma_4}{4h^5}\sum_{\substack{p,q\\p\neq q}}^{n}\Big[e_p\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T + e_r\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\mu_r\delta_r\mu_p\delta_p\tilde{\mathbf{h}}^T \\
& + e_q\delta_p^2\tilde{\mathbf{h}}^T\delta_p^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\Big] \\
+ \quad & \frac{\sigma_2^3}{4h^5}\sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n}e_r\Big[\mu_r\delta_r\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T + \mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T + \delta_p^2\tilde{\mathbf{h}}^T\delta_q^2\tilde{\mathbf{h}}\mu_r\delta_r\tilde{\mathbf{h}}^T\Big] \\
+ \quad & \frac{\sigma_2\sigma_4}{4h^5}\sum_{\substack{p,q\\p\neq q}}^{n}\Big[e_q\delta_p^2\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T + e_p\delta_p^2\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T \\
& + e_q\delta_q^2\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T + e_q\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\delta_p^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\Big] \\
+ \quad & \frac{\sigma_2\sigma_4}{4h^5}\sum_{\substack{p,q\\p\neq q}}^{n}\Big[e_p\delta_q^2\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T + e_q\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\delta_p^2\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\Big] \\
+ \quad & \frac{\sigma_2\sigma_4}{4h^5}\sum_{\substack{p,q\\p\neq q}}^{n}\Big[e_q\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\delta_q^2\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T + e_p\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\delta_q^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\Big] \\
+ \quad & \frac{\sigma_2^3}{4h^5}\sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n}e_r\Big[\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\delta_r^2\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T + \mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\delta_r^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\Big]
\end{aligned}
$$

$$
\begin{aligned}
L_{i,j} &= \frac{\sigma_4-\sigma_2^2}{2h^4}\sum_{p=1}^{n}\left[\mu_p\delta_p\tilde{\mathbf{h}}_i\mu_p\delta_p\tilde{\mathbf{h}}_j\delta_p^2\tilde{\mathbf{h}}^T+\mu_p\delta_p\tilde{\mathbf{h}}_i\delta_p^2\tilde{\mathbf{h}}_j\mu_p\delta_p\tilde{\mathbf{h}}^T+\delta_p^2\tilde{\mathbf{h}}_i\mu_p\delta_p\tilde{\mathbf{h}}_j\mu_p\delta_p\tilde{\mathbf{h}}^T\right]\\
&\quad+\frac{\sigma_2^2}{2h^4}\sum_{\substack{p,q=1\\p\neq q}}^{n}\mu_p\delta_p\tilde{\mathbf{h}}_i\mu_q\delta_q\tilde{\mathbf{h}}_j\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T+\frac{\sigma_2}{2h^2}\sum_{p=1}^{n}\delta_p^2\tilde{\mathbf{h}}^T\delta_{i,j}R_{i,j}\\
&+\frac{\sigma_2^2}{2h^4}\sum_{\substack{p,q=1\\p\neq q}}^{n}\left[\mu_p\delta_p\tilde{\mathbf{h}}_i\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}_j\mu_q\delta_q\tilde{\mathbf{h}}^T+\mu_q\delta_q\tilde{\mathbf{h}}_i\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}_j\mu_p\delta_p\tilde{\mathbf{h}}^T\right]\\
&+\frac{\sigma_2}{2h^2}\sum_{p=1}^{n}\delta_p^2\tilde{\mathbf{h}}_ie_{j^T}R_{j,j}+\frac{\sigma_2^2}{2h^4}\sum_{\substack{p,q=1\\p\neq q}}^{n}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}_i\mu_p\delta_p\tilde{\mathbf{h}}_j\mu_q\delta_q\tilde{\mathbf{h}}^T\\
&+\frac{\sigma_2^2}{2h^4}\sum_{\substack{p,q=1\\p\neq q}}^{n}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}_i\mu_q\delta_q\tilde{\mathbf{h}}_j\mu_p\delta_p\tilde{\mathbf{h}}^T+\frac{\sigma_6+2\sigma_2^3-3\sigma_2\sigma_4}{8h^6}\sum_{p=1}^{n}(\delta_p^2\tilde{\mathbf{h}}_i\delta_p^2\tilde{\mathbf{h}}_j\delta_p^2\tilde{\mathbf{h}}^T)\\
&+\frac{\sigma_4\sigma_2-\sigma_2^3}{8h^6}\sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n}\delta_p^2\tilde{\mathbf{h}}_i\delta_q^2\tilde{\mathbf{h}}_j\delta_r^2\tilde{\mathbf{h}}^T+\frac{\sigma_4\sigma_2}{8h^6}\sum_{\substack{p,q=1\\p\neq q}}^{n}\delta_q^2\tilde{\mathbf{h}}_i\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}_j\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}^T\\
&\quad+\frac{\sigma_4\sigma_2-\sigma_2^3}{8h^6}\sum_{\substack{p,q=1\\p\neq q}}^{n}\left[\delta_p^2\tilde{\mathbf{h}}_i\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}_j\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T+\delta_q^2\tilde{\mathbf{h}}_i\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}_j\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\right]\\
&\quad+\frac{\sigma_4\sigma_2-\sigma_2^3}{8h^6}\sum_{\substack{p,q=1\\p\neq q}}^{n}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}_i\delta_p^2\tilde{\mathbf{h}}_j\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T+\frac{\sigma_2}{2h^2}\sum_{p=1}^{n}\delta_p^2\tilde{\mathbf{h}}_je_iR_{i,i}\\
&+\frac{\sigma_4\sigma_2}{8h^6}\sum_{\substack{p,q=1\\p\neq q}}^{n}\left[\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}_i\delta_p^2\tilde{\mathbf{h}}_j\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}^T+\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}_i\delta_q^2\tilde{\mathbf{h}}_j\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}^T\right]\\
&\quad+\frac{\sigma_4\sigma_2-\sigma_2^3}{8h^6}\sum_{\substack{p,q=1\\p\neq q}}^{n}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}_i\delta_q^2\tilde{\mathbf{h}}_j\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\\
&\quad+\frac{\sigma_4\sigma_2-\sigma_2^3}{8h^6}\sum_{\substack{p,q=1\\p\neq q}}^{n}\left(\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}_i\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}_j\delta_p^2\tilde{\mathbf{h}}^T+\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}_i\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}_j\delta_q^2\tilde{\mathbf{h}}^T\right)\\
&+\frac{\sigma_2^3}{8h^6}\sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n}\Big(\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}_i\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}_j\mu_p\delta_p\mu_r\delta_r\tilde{\mathbf{h}}^T+\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}_i\delta_r^2\tilde{\mathbf{h}}_j\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}^T\\
&\quad+\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}_i\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}_j\mu_r\delta_r\mu_p\delta_p\tilde{\mathbf{h}}^T\Big)+\frac{\sigma_2}{4h^2}\sum_{p=1}^{n}\delta_p^2h_ie_j^TR_{j,j}+\delta_p^2h_je_i^TR_{i,i}
\end{aligned}
$$

$$
\begin{aligned}
+ &\quad \frac{\sigma_2^3}{8h^6} \sum_{\substack{p,q,r=1 \\ p \neq q \neq r}}^{n} \Big(\mu_p \delta_p \mu_q \delta_q \tilde{\mathbf{h}}_i \mu_p \delta_p \mu_r \delta_r \tilde{\mathbf{h}}_j \mu_q \delta_q \mu_r \delta_r \tilde{\mathbf{h}}^T \\
&\quad + \mu_p \delta_p \mu_q \delta_q \tilde{\mathbf{h}}_i \mu_p \delta_p \mu_r \delta_r \tilde{\mathbf{h}}_j \mu_r \delta_r \mu_q \delta_q \tilde{\mathbf{h}}^T \Big) - \frac{\sigma_2}{2h^2} \sum_{p=1}^{n} \delta_p^2 \tilde{\mathbf{h}}_i \left(e_j^T R_{j,j} \right) \\
+ &\quad \frac{\sigma_2^2}{8h^6} \sum_{\substack{p,q=1 \\ p \neq q}}^{n} \mu_p \delta_p \tilde{\mathbf{h}}_i \mu_q \delta_q \tilde{\mathbf{h}}_j \mu_q \delta_q \mu_p \delta_p \tilde{\mathbf{h}}^T \\
&\quad + \frac{\sigma_2^3}{8h^6} \sum_{\substack{p,q,r=1 \\ p \neq q \neq r}}^{n} \Big[\delta_p^2 \tilde{\mathbf{h}}_i \mu_q \delta_q \mu_r \delta_r \tilde{\mathbf{h}}_j \mu_r \delta_r \mu_q \delta_q \tilde{\mathbf{h}}^T + \mu_p \delta_p \mu_q \delta_q \tilde{\mathbf{h}}_i \mu_q \delta_q \mu_p \delta_p \tilde{\mathbf{h}}_j \delta_r^2 \tilde{\mathbf{h}}^T \Big] \\
+ &\quad \frac{\sigma_4 \sigma_2}{8h^6} \sum_{\substack{p,q,r=1 \\ p \neq q \neq r}}^{n} \Big[\delta_p^2 \tilde{\mathbf{h}}_i \mu_p \delta_p \mu_q \delta_q \tilde{\mathbf{h}}_j \mu_q \delta_q \mu_p \delta_p \tilde{\mathbf{h}}^T + \mu_p \delta_p \mu_q \delta_q \tilde{\mathbf{h}}_i \mu_q \delta_q \mu_p \delta_p \tilde{\mathbf{h}}_j \delta_p^2 \tilde{\mathbf{h}}^T \\
&\quad + \mu_p \delta_p \mu_q \delta_q \tilde{\mathbf{h}}_i \mu_q \delta_q \mu_p \delta_p \tilde{\mathbf{h}}_j \delta_q^2 \tilde{\mathbf{h}}^T \Big]
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}\left[\mathcal{A}\mathcal{A}^T\mathcal{A}\mathcal{A}^T | Y_{t_{k-1}}\right] = \frac{1}{4h^6} \Bigg(\sum_{p=1}^{n} \sigma_6 \delta_p^2 \tilde{\mathbf{h}} \mu_p \delta_p \tilde{\mathbf{h}}^T \delta_p^2 \tilde{\mathbf{h}} \mu_p \delta_p \tilde{\mathbf{h}}^T \\
&\quad + \sum_{\substack{p,q,r=1 \\ p \neq q \neq r}}^{n} \sigma_2^3 \mu_p \delta_p \mu_q \delta_q \tilde{\mathbf{h}} \mu_p \delta_p \mu_q \delta_q \tilde{\mathbf{h}}^T \mu_r \delta_r \tilde{\mathbf{h}} \mu_r \delta_r \tilde{\mathbf{h}}^T \\
+ &\quad \sum_{\substack{p,q=1 \\ p \neq q}}^{n} \sigma_2 \sigma_4 \Big[\delta_q^2 \tilde{\mathbf{h}} \mu_q \delta_q \tilde{\mathbf{h}}^T \delta_p^2 \tilde{\mathbf{h}}(t_k, \hat{z}(t_k)) \mu_q \delta_q \tilde{\mathbf{h}}^T + \delta_q^2 \tilde{\mathbf{h}} \mu_q \delta_q \tilde{\mathbf{h}}^T \mu_p \delta_p \mu_q \delta_q \tilde{\mathbf{h}} \mu_p \delta_p \tilde{\mathbf{h}}^T \\
&\quad + \delta_p^2 \tilde{\mathbf{h}} \mu_q \delta_q \tilde{\mathbf{h}}^T \delta_q^2 \tilde{\mathbf{h}} \mu_q \delta_q \tilde{\mathbf{h}}^T + \mu_p \delta_p \mu_q \delta_q \tilde{\mathbf{h}} \mu_p \delta_p \tilde{\mathbf{h}}^T \delta_q^2 \tilde{\mathbf{h}} \mu_q \delta_q \tilde{\mathbf{h}}^T \Big] \\
+ &\quad \sum_{\substack{p,q=1 \\ p \neq q}}^{n} \sigma_2 \sigma_4 \Big[\delta_p^2 \tilde{\mathbf{h}} \mu_q \delta_q \tilde{\mathbf{h}}^T \delta_p^2 \tilde{\mathbf{h}} \mu_q \delta_q \tilde{\mathbf{h}}^T + \delta_p^2 \tilde{\mathbf{h}} \mu_q \delta_q \tilde{\mathbf{h}}^T \mu_p \delta_p \mu_q \delta_q \tilde{\mathbf{h}} \mu_p \delta_p \tilde{\mathbf{h}}^T \Big] \\
+ &\quad \sum_{\substack{p,q,r=1 \\ p \neq q \neq r}}^{n} \sigma_2^3 \Big[\delta_p^2 \tilde{\mathbf{h}} \mu_q \delta_q \tilde{\mathbf{h}}^T \mu_r \delta_r \mu_q \delta_q \tilde{\mathbf{h}} \mu_r \delta_r \tilde{\mathbf{h}}^T + \delta_p^2 \tilde{\mathbf{h}} \mu_q \delta_q \tilde{\mathbf{h}}^T \mu_q \delta_q \mu_r \delta_r \tilde{\mathbf{h}} \mu_r \delta_r \tilde{\mathbf{h}}^T \\
&\quad + \mu_r \delta_r \mu_q \delta_q \tilde{\mathbf{h}} \mu_r \delta_r \tilde{\mathbf{h}}^T \delta_p^2 \tilde{\mathbf{h}} \mu_q \delta_q \tilde{\mathbf{h}}^T \Big] \\
+ &\quad \sum_{\substack{p,q=1 \\ p \neq q}}^{n} \sigma_2 \sigma_4 \Big[\mu_p \delta_p \mu_q \delta_q \tilde{\mathbf{h}} \mu_p \delta_p \tilde{\mathbf{h}}^T \delta_p^2 \tilde{\mathbf{h}} \mu_q \delta_q \tilde{\mathbf{h}}^T + \mu_p \delta_p \mu_q \delta_q \tilde{\mathbf{h}} \mu_q \delta_q \tilde{\mathbf{h}}^T \delta_p^2 \tilde{\mathbf{h}} \mu_q \delta_q \tilde{\mathbf{h}}^T \\
&\quad + \mu_p \delta_p \mu_q \delta_q \tilde{\mathbf{h}} \mu_p \delta_p \tilde{\mathbf{h}}^T \mu_p \delta_p \mu_q \delta_q \tilde{\mathbf{h}} \mu_p \delta_p \tilde{\mathbf{h}}^T \Big]
\end{aligned}
$$

$$
\begin{aligned}
+ & \sum_{\substack{p,q,r=1\\ p\neq q\neq r}}^{n} \sigma_2^3 \Big[\delta_p^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\delta_r^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T + \mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}\mu_r\delta_r\tilde{\mathbf{h}}^T\delta_p^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T \\
& +\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}\mu_r\delta_r\tilde{\mathbf{h}}^T\Big] \\
+ & \sum_{\substack{p,q=1\\ p\neq q}}^{n} \sigma_2\sigma_4 \Big[\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T \\
& +\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T + \mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\delta_p^2\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\Big] \\
+ & \sum_{\substack{p,q=1\\ p\neq q}}^{n} \sigma_2\sigma_4 \Big[\delta_p^2\tilde{\mathbf{h}}(t_k,\hat{z}(t_k))\mu_p\delta_p\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T + \delta_p^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T \\
& +\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\Big] \\
+ & \sum_{\substack{p,q,r=1\\ p\neq q\neq r}}^{n} \sigma_2^3 \Big[\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\mu_r\delta_r\mu_q\delta_q\tilde{\mathbf{h}}\mu_r\delta_r\tilde{\mathbf{h}}^T \\
& +\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_r\delta_r\tilde{\mathbf{h}}\mu_r\delta_r\tilde{\mathbf{h}}^T + \mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}\mu_r\delta_r\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\Big] \\
+ & \sum_{\substack{p,q,r=1\\ p\neq q\neq r}}^{n} \sigma_2^3\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_r\delta_r\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_r\delta_r\tilde{\mathbf{h}}^T + \sum_{p=1}^{n}\sigma_6\delta_p^2\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T \\
+ & \sum_{\substack{p,q=1\\ p\neq q}}^{n} \sigma_2\sigma_4 \Big[\delta_q^2\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T + \delta_q^2\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T \\
& +\delta_p^2\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\Big] \\
+ & \sum_{\substack{p,q=1\\ p\neq q}}^{n} \sigma_2\sigma_4 \Big[\delta_p^2\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T + \delta_p^2\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T \\
& +\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\Big] \\
+ & \sum_{\substack{p,q,r=1\\ p\neq q\neq r}}^{n} \sigma_2^3 \Big[\delta_p^2\tilde{\mathbf{h}}\mu_r\delta_r\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\mu_r\delta_r\tilde{\mathbf{h}}^T + \delta_p^2\tilde{\mathbf{h}}\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\mu_r\delta_r\tilde{\mathbf{h}}^T \\
& +\mu_r\delta_r\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\mu_r\delta_r\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\Big]
\end{aligned}
$$

$$
\begin{aligned}
+ & \sum_{\substack{p,q=1\\p\neq q}}^{n} \sigma_2\sigma_4 \Big[\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T + \mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T \\
& +\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\Big] \\
+ & \sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n} \sigma_2^3 \Big[\delta_p^2\tilde{\mathbf{h}}\delta_r^2\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T + \mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\mu_r\delta_r\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T \\
& +\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\mu_r\delta_r\tilde{\mathbf{h}}^T\Big] \\
+ & \sum_{\substack{p,q=1\\p\neq q}}^{n} \sigma_2\sigma_4 \Big[\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T \\
& +\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T + \mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\Big] \\
+ & \sum_{\substack{p,q=1\\p\neq q}}^{n} \sigma_2\sigma_4 \Big[\delta_p^2\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T + \delta_p^2\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T \\
& +\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\Big] \\
+ & \sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n} \sigma_2^3 \Big[\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_r\delta_r\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\mu_r\delta_r\tilde{\mathbf{h}}^T \\
& +\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_r\delta_r\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\mu_r\delta_r\tilde{\mathbf{h}}^T + \mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_r\delta_r\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\Big] \\
+ & \sum_{p=1}^{n}\sigma_6\mu_p\delta_p\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\delta_p^2\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T + \sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n}\sigma_2^3\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_r\delta_r\tilde{\mathbf{h}}^T\mu_r\delta_r\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T \\
+ & \sum_{\substack{p,q=1\\p\neq q}}^{n} \sigma_2\sigma_4 \Big[\mu_q\delta_q\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T\delta_p^2\tilde{\mathbf{h}}(t_k,\hat{z}(t_k))\mu_q\delta_q\tilde{\mathbf{h}}^T + \mu_q\delta_q\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T \\
& +\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\delta_q^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\Big] \\
+ & \sum_{\substack{p,q=1\\p\neq q}}^{n} \sigma_2\sigma_4 \Big[\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\delta_p^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T + \mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T \\
& +\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\delta_q^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\Big] \\
+ & \sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n} \sigma_2^3 \Big[\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\mu_r\delta_r\mu_q\delta_q\tilde{\mathbf{h}}\mu_r\delta_r\tilde{\mathbf{h}}^T + \mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}\mu_r\delta_r\tilde{\mathbf{h}}^T \\
& +\mu_r\delta_r\tilde{\mathbf{h}}\mu_r\delta_r\mu_q\delta_q\tilde{\mathbf{h}}^T\delta_p^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\Big]
\end{aligned}
$$

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\begin{aligned}
+ & \sum_{\substack{p,q=1 \\ p\neq q}}^{n} \sigma_2\sigma_4 \Big[\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\delta_p^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T + \mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\delta_p^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T \\
& +\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\Big] \\
+ & \sum_{\substack{p,q,r=1 \\ p\neq q\neq r}}^{n} \sigma_2^3 \Big[\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\delta_r^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T + \mu_r\delta_r\tilde{\mathbf{h}}\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}^T\delta_p^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T \\
& +\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}\mu_r\delta_r\tilde{\mathbf{h}}^T\Big] \\
+ & \sum_{\substack{p,q=1 \\ p\neq q}}^{n} \sigma_2\sigma_4 \Big[\mu_p\delta_p\tilde{\mathbf{h}}\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T \\
& +\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T + \mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\delta_p^2\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\Big] \\
+ & \sum_{\substack{p,q=1 \\ p\neq q}}^{n} \sigma_2\sigma_4 \Big[\mu_p\delta_p\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T + \mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T \\
& +\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T + \mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\Big] \\
+ & \sum_{\substack{p,q,r=1 \\ p\neq q\neq r}}^{n} \sigma_2^3 \Big[\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_r\delta_r\mu_q\delta_q\tilde{\mathbf{h}}\mu_r\delta_r\tilde{\mathbf{h}}^T \\
& +\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_r\delta_r\tilde{\mathbf{h}}\mu_r\delta_r\tilde{\mathbf{h}}^T + \mu_r\delta_r\tilde{\mathbf{h}}\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\Big] \\
+ & \sum_{\substack{p,q,r=1 \\ p\neq q\neq r}}^{n} \sigma_2^3\mu_r\delta_r\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_r\delta_r\tilde{\mathbf{h}}^T + \sum_{p=1}^{n}\sigma_6\delta_p^2\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T \\
+ & \sum_{\substack{p,q=1 \\ p\neq q}}^{n} \sigma_2\sigma_4 \Big[\delta_q^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T + \delta_q^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\Big] \\
+ & \sum_{\substack{p,q=1 \\ p\neq q}}^{n} \sigma_2\sigma_4 \Big[\delta_p^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T + \delta_p^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T \\
& +\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T + \delta_p^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T\Big] \\
+ & \sum_{\substack{p,q,r=1 \\ p\neq q\neq r}}^{n} \sigma_2^3 \Big[\delta_p^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_r\delta_r\tilde{\mathbf{h}}\mu_r\delta_r\mu_q\delta_q\tilde{\mathbf{h}}^T + \delta_p^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_r\delta_r\tilde{\mathbf{h}}\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}^T \\
& +\mu_r\delta_r\mu_q\delta_q\tilde{\mathbf{h}}\mu_r\delta_r\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\Big] \\
+ & \sum_{\substack{p,q=1 \\ p\neq q}}^{n} \sigma_2\sigma_4 \Big[\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T + \mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\Big]
\end{aligned}
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\begin{aligned}
&+ \sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n} \sigma_2^3 \Big[\delta_p^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\delta_r^2\tilde{\mathbf{h}}^T + \mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}\mu_r\delta_r\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T \\
&\quad +\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\mu_r\delta_r\tilde{\mathbf{h}}\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}^T\Big] \\
&+ \sum_{\substack{p,q=1\\p\neq q}}^{n} \sigma_2\sigma_4 \Big[\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T \\
&\quad +\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T + \mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\Big] \\
&+ \sum_{\substack{p,q=1\\p\neq q}}^{n} \sigma_2\sigma_4 \Big[\delta_p^2\tilde{\mathbf{h}}(t_k,\hat{z}(t_k))\mu_p\delta_p\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T + \delta_p^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T \\
&\quad +\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}^T\Big] \\
&+ \sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n} \sigma_2^3 \Big[\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\mu_r\delta_r\tilde{\mathbf{h}}\mu_r\delta_r\mu_q\delta_q\tilde{\mathbf{h}}^T \\
&\quad +\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_r\delta_r\tilde{\mathbf{h}}\mu_p\delta_p\mu_r\delta_r\tilde{\mathbf{h}}^T + \mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}\mu_r\delta_r\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\Big] \\
&+ \sum_{p=1}^{n}\sigma_6\mu_p\delta_p\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T + \sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n} \sigma_2^3\mu_r\delta_r\tilde{\mathbf{h}}\mu_r\delta_r\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T \\
&+ \sum_{\substack{p,q=1\\p\neq q}}^{n} \sigma_2\sigma_4 \Big[\mu_q\delta_q\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T + \mu_q\delta_q\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T \\
&\quad +\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T\Big] \\
&+ \sum_{\substack{p,q=1\\p\neq q}}^{n} \sigma_2\sigma_4 \Big[\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T + \mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T \\
&\quad +\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T\Big] \\
&+ \sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n} \sigma_2^3 \Big[\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\mu_r\delta_r\tilde{\mathbf{h}}\mu_r\delta_r\mu_q\delta_q\tilde{\mathbf{h}}^T + \mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\mu_r\delta_r\tilde{\mathbf{h}}\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}^T \\
&\quad +\mu_r\delta_r\tilde{\mathbf{h}}\mu_r\delta_r\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\Big] \\
&+ \sum_{\substack{p,q=1\\p\neq q}}^{n} \sigma_2\sigma_4 \Big[\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T + \mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T \\
&\quad +\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\Big]
\end{aligned}
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\begin{aligned}
&+ \sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n} \sigma_2^3 \Big[\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\delta_r^2\tilde{\mathbf{h}}^T + \mu_r\delta_r\tilde{\mathbf{h}}\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T \\
&\quad +\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_r\delta_r\tilde{\mathbf{h}}\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}^T\Big] \\
&+ \sum_{\substack{p,q=1\\p\neq q}}^{n} \sigma_2\sigma_4 \Big[\mu_p\delta_p\tilde{\mathbf{h}}\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T \\
&\quad +\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T + \mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\Big] \\
&+ \sum_{\substack{p,q=1\\p\neq q}}^{n} \sigma_2\sigma_4 \Big[\mu_p\delta_p\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T + \mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T \\
&\quad +\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}^T\Big] \\
&+ \sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n} \sigma_2^3 \Big[\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_r\delta_r\tilde{\mathbf{h}}\mu_r\delta_r\mu_q\delta_q\tilde{\mathbf{h}}^T \\
&\quad +\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_r\delta_r\tilde{\mathbf{h}}\mu_p\delta_p\mu_r\delta_r\tilde{\mathbf{h}}^T + \mu_r\delta_r\tilde{\mathbf{h}}\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\Big] \\
&+ \sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n} \sigma_2^3\mu_r\delta_r\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_r\delta_r\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T + \sum_{p=1}^{n}\sigma_6\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\delta_p^2\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T \\
&+ \sum_{\substack{p,q=1\\p\neq q}}^{n} \sigma_2\sigma_4 \Big[\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\delta_q^2\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T + \mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\delta_q^2\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T \\
&\quad +\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\delta_p^2\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T\Big] \\
&+ \sum_{\substack{p,q=1\\p\neq q}}^{n} \sigma_2\sigma_4 \Big[\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\delta_p^2\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T + \mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\delta_p^2\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T \\
&\quad +\mu_p\delta_p\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T\Big] \\
&+ \sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n} \sigma_2^3 \Big[\mu_q\delta_q\tilde{\mathbf{h}}\mu_r\delta_r\tilde{\mathbf{h}}^T\delta_p^2\tilde{\mathbf{h}}\mu_r\delta_r\mu_q\delta_q\tilde{\mathbf{h}}^T + \mu_q\delta_q\tilde{\mathbf{h}}\mu_r\delta_r\tilde{\mathbf{h}}^T\delta_p^2\tilde{\mathbf{h}}\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}^T \\
&\quad +\mu_r\delta_r\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_r\delta_r\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\Big] \\
&+ \sum_{\substack{p,q=1\\p\neq q}}^{n} \sigma_2\sigma_4 \Big[\mu_p\delta_p\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T + \mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T \\
&\quad +\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\Big]
\end{aligned}
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$$
\begin{aligned}
+ & \sum_{\substack{p,q,r=1\\ p\neq q\neq r}}^{n} \sigma_2^3 \Big[\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\delta_p^2\tilde{\mathbf{h}}\delta_r^2\tilde{\mathbf{h}}^T + \mu_r\delta_r\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T \\
& +\mu_p\delta_p\tilde{\mathbf{h}}\mu_r\delta_r\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}^T\Big] \\
+ & \sum_{\substack{p,q=1\\ p\neq q}}^{n} \sigma_2\sigma_4 \Big[\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T \\
& +\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T + \mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\Big] \\
+ & \sum_{\substack{p,q=1\\ p\neq q}}^{n} \sigma_2\sigma_4 \Big[\mu_p\delta_p\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\delta_p^2\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T + \mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\delta_p^2\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T \\
& +\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}^T\Big] \\
+ & \sum_{\substack{p,q,r=1\\ p\neq q\neq r}}^{n} \sigma_2^3 \Big[\mu_p\delta_p\tilde{\mathbf{h}}\mu_r\delta_r\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_r\delta_r\mu_q\delta_q\tilde{\mathbf{h}}^T \\
& +\mu_q\delta_q\tilde{\mathbf{h}}\mu_r\delta_r\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_r\delta_r\tilde{\mathbf{h}}^T + \mu_r\delta_r\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\Big]\Big) \\
+ & \frac{\sigma_2 R}{h^2}\sum_{p=1}^{n}\Big[2\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T + \mu_p\delta_p\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\Big] \\
& +\frac{2R}{4h^4}\left[\sum_{p=1}^{n}\sigma_4\delta_p^2\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T + \sigma_2^2\sum_{\substack{p,q=1\\ p\neq q}}^{n}\delta_p^2\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T + \mu_p\delta_p\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\right] \\
& +\frac{\sigma_2}{h^2}\sum_{p=1}^{n}\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\left[\sum_{i=1}^{n}(I_{n,n}R_{i,i}) + 2R\right] \\
+ & \frac{R}{4h^4}\left[\sum_{p=1}^{n}\sigma_4\delta_p^2\tilde{\mathbf{h}}^T\delta_p^2\tilde{\mathbf{h}} + \sigma_2^2\sum_{\substack{p,q=1\\ p\neq q}}^{n}\delta_p^2\tilde{\mathbf{h}}^T\delta_q^2\tilde{\mathbf{h}} + \mu_p\delta_p\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\right] \\
+ & \frac{2}{4h^4}\left[\sum_{p=1}^{n}\sigma_4\delta_p^2\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T + \sigma_2^2\sum_{\substack{p,q=1\\ p\neq q}}^{n}\delta_p^2\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T + \mu_p\delta_p\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\right] R \\
& +3RR^T.
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}\left[\mathcal{A}\mathcal{A}^T\mathcal{A}\mathcal{C}^T|Y_{t_{k-1}}\right] = \\
&S_x\left(\frac{\sigma_4}{2h^4}\sum_{p=1}^{n}\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\delta_p^2\tilde{\mathbf{h}}+\frac{\sigma_2^2}{2h^4}\sum_{\substack{p,q=1\\p\neq q}}^{n}\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\delta_q^2\tilde{\mathbf{h}}\right.\\
&+\frac{\sigma_2^2}{2h^4}\sum_{\substack{p,q=1\\p\neq q}}^{n}\mu_p\delta_p\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}+\frac{\sigma_4}{2h^4}\sum_{p=1}^{n}\mu_p\delta_p\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\\
&+\frac{\sigma_2^2}{2h^4}\sum_{\substack{p,q=1\\p\neq q}}^{n}(\mu_p\delta_p\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}})+\frac{\sigma_2^2}{2h^4}\sum_{\substack{p,q=1\\p\neq q}}^{n}(\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}})\\
+\quad&\frac{\sigma_2^2}{2h^4}\sum_{\substack{p,q=1\\p\neq q}}^{n}(\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}})+\frac{\sigma_4}{2h^4}\sum_{p=1}^{n}\delta_p^2\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\\
&+\frac{\sigma_2^2}{2h^4}\sum_{\substack{p,q=1\\p\neq q}}^{n}\delta_p^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}})+\frac{\sigma_2^2}{2h^4}\sum_{\substack{p,q=1\\p\neq q}}^{n}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\\
&+\frac{\sigma_2^2}{2h^4}\sum_{\substack{p,q=1\\p\neq q}}^{n}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}+\frac{\sigma_6}{8h^6}\sum_{p=1}^{n}(\delta_p^2\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\delta_p^2\tilde{\mathbf{h}})\\
+\quad&\frac{\sigma_4\sigma_2}{8h^6}\sum_{\substack{p,q=1\\p\neq q}}^{n}\delta_p^2\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\delta_q^2\tilde{\mathbf{h}}+\frac{\sigma_4\sigma_2}{8h^6}\sum_{\substack{p,q=1\\p\neq q}}^{n}\delta_p^2\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T\delta_p^2\tilde{\mathbf{h}}\\
&+\frac{\sigma_4\sigma_2}{8h^6}\sum_{\substack{p,q=1\\p\neq q}}^{n}\delta_p^2\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T\delta_q^2\tilde{\mathbf{h}}+\frac{\sigma_4\sigma_2}{8h^6}\sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n}\delta_p^2\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T\delta_r^2\tilde{\mathbf{h}}\\
&+\frac{\sigma_4\sigma_2}{8h^6}\sum_{\substack{p,q=1\\p\neq q}}^{n}\delta_p^2\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}+\frac{\sigma_4\sigma_2}{8h^6}\sum_{\substack{p,q=1\\p\neq q}}^{n}\delta_q^2\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\\
+\quad&\frac{\sigma_4\sigma_2}{8h^6}\sum_{\substack{p,q=1\\p\neq q}}^{n}\delta_q^2\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}+\frac{\sigma_2^3}{8h^6}\sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n}\delta_p^2\tilde{\mathbf{h}}\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}^T\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}\\
&+\frac{\sigma_4\sigma_2}{8h^6}\sum_{\substack{p,q=1\\p\neq q}}^{n}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\\
+\quad&\frac{\sigma_4\sigma_2}{8h^6}\sum_{\substack{p,q=1\\p\neq q}}^{n}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}+\frac{\sigma_4\sigma_2}{8h^6}\sum_{\substack{p,q=1\\p\neq q}}^{n}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\\
&+\frac{\sigma_4\sigma_2}{8h^6}\sum_{\substack{p,q=1\\p\neq q}}^{n}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}+\frac{\sigma_2^3}{8h^6}\sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\delta_r^2\tilde{\mathbf{h}}
\end{aligned}
$$

$$+ \quad \frac{\sigma_2^3}{8h^6} \sum_{\substack{p,q,r=1 \\ p \neq q \neq r}}^{n} \mu_p \delta_p \mu_q \delta_q \tilde{\mathbf{h}} \delta_r^2 \tilde{\mathbf{h}}^T \mu_p \delta_p \mu_q \delta_q \tilde{\mathbf{h}} + \frac{\sigma_2^3}{8h^6} \sum_{\substack{p,q,r=1 \\ p \neq q \neq r}}^{n} \mu_p \delta_p \mu_q \delta_q \tilde{\mathbf{h}} \delta_r^2 \tilde{\mathbf{h}}^T \mu_q \delta_q \mu_p \delta_p \tilde{\mathbf{h}}$$

$$+ \quad \frac{\sigma_4 \sigma_2}{8h^6} \sum_{\substack{p,q=1 \\ p \neq q}}^{n} \Big(\mu_p \delta_p \mu_q \delta_q \tilde{\mathbf{h}} \mu_p \delta_p \mu_q \delta_q \tilde{\mathbf{h}}^T \delta_p^2 \tilde{\mathbf{h}} + \mu_p \delta_p \mu_q \delta_q \tilde{\mathbf{h}} \mu_p \delta_p \mu_q \delta_q \tilde{\mathbf{h}}^T \delta_q^2 \tilde{\mathbf{h}} \Big)$$

$$+ \quad \frac{\sigma_2^3}{8h^6} \sum_{\substack{p,q,r=1 \\ p \neq q \neq r}}^{n} \Big(\mu_p \delta_p \mu_q \delta_q \tilde{\mathbf{h}} \mu_q \delta_q \mu_r \delta_r \tilde{\mathbf{h}}^T \mu_p \delta_p \mu_r \delta_r \tilde{\mathbf{h}} + \mu_p \delta_p \mu_q \delta_q \tilde{\mathbf{h}} \mu_q \delta_q \mu_r \delta_r \tilde{\mathbf{h}}^T \mu_r \delta_r \mu_p \delta_p \tilde{\mathbf{h}} \Big)$$

$$+ \quad \frac{\sigma_2^3}{8h^6} \sum_{\substack{p,q,r=1 \\ p \neq q \neq r}}^{n} \Big(\mu_p \delta_p \mu_q \delta_q \tilde{\mathbf{h}} \mu_p \delta_p \mu_r \delta_r \tilde{\mathbf{h}}^T \mu_q \delta_q \mu_r \delta_r \tilde{\mathbf{h}} + \mu_p \delta_p \mu_q \delta_q \tilde{\mathbf{h}} \mu_p \delta_p \mu_r \delta_r \tilde{\mathbf{h}}^T \mu_r \delta_r \mu_q \delta_q \tilde{\mathbf{h}} \Big)$$

$$\left. + \frac{\sigma_2}{4h^2} \sum_{p=1}^{n} \left[\delta_p^2 h_i e_j R_{j,j} + \delta_p^2 h_j e_i R_{i,i} \right] \right) C^T + \frac{\sigma_2}{2h^2} \sum_{p=1}^{n} \delta_p^2 \tilde{\mathbf{h}} \delta_{i,j} R_{i,j}$$

$$\mathbb{E}\left[\mathcal{A}\mathcal{A}^T C \mathcal{A}^T | Y_{t_{k-1}} \right] =$$

$$S_x \left(\frac{\sigma_4}{2h^4} \sum_{p=1}^{n} \mu_p \delta_p \tilde{\mathbf{h}} \mu_p \delta_p \tilde{\mathbf{h}}^T C \delta_p^2 \tilde{\mathbf{h}}^T + \frac{\sigma_2^2}{2h^4} \sum_{\substack{p,q=1 \\ p \neq q}}^{n} \mu_p \delta_p \tilde{\mathbf{h}} \mu_p \delta_p \tilde{\mathbf{h}}^T C \delta_q^2 \tilde{\mathbf{h}}^T \right.$$

$$+ \frac{\sigma_2^2}{2h^4} \sum_{\substack{p,q=1 \\ p \neq q}}^{n} \mu_p \delta_p \tilde{\mathbf{h}} \mu_q \delta_q \tilde{\mathbf{h}}^T C \mu_p \delta_p \mu_q \delta_q \tilde{\mathbf{h}}^T + \frac{\sigma_4}{2h^4} \sum_{p=1}^{n} \mu_p \delta_p \tilde{\mathbf{h}} \delta_p^2 \tilde{\mathbf{h}}^T C \mu_p \delta_p \tilde{\mathbf{h}}^T$$

$$+ \frac{\sigma_2^2}{2h^4} \sum_{\substack{p,q=1 \\ p \neq q}}^{n} \mu_p \delta_p \tilde{\mathbf{h}} \delta_q^2 \tilde{\mathbf{h}}^T C \mu_p \delta_p \tilde{\mathbf{h}}^T + \mu_p \delta_p \tilde{\mathbf{h}} \mu_p \delta_p \mu_q \delta_q \tilde{\mathbf{h}}^T C \mu_q \delta_q \tilde{\mathbf{h}}^T$$

$$+ \quad \frac{\sigma_2^2}{2h^4} \sum_{\substack{p,q=1 \\ p \neq q}}^{n} (\mu_q \delta_q \tilde{\mathbf{h}} \mu_p \delta_p \mu_q \delta_q \tilde{\mathbf{h}}^T C \mu_p \delta_p \tilde{\mathbf{h}}^T) + \frac{\sigma_4}{2h^4} \sum_{p=1}^{n} \delta_p^2 \tilde{\mathbf{h}} \mu_p \delta_p \tilde{\mathbf{h}}^T C \mu_p \delta_p \tilde{\mathbf{h}}^T$$

$$+ \frac{\sigma_2^2}{2h^4} \sum_{\substack{p,q=1 \\ p \neq q}}^{n} \Big[\delta_p^2 \tilde{\mathbf{h}} \mu_q \delta_q \tilde{\mathbf{h}}^T C \mu_q \delta_q \tilde{\mathbf{h}}^T + \mu_p \delta_p \mu_q \delta_q \tilde{\mathbf{h}} \mu_p \delta_p \tilde{\mathbf{h}}^T C \mu_q \delta_q \tilde{\mathbf{h}}^T \Big]$$

$$+ \quad \frac{\sigma_2^2}{2h^4} \sum_{\substack{p,q=1 \\ p \neq q}}^{n} \mu_p \delta_p \mu_q \delta_q \tilde{\mathbf{h}} \mu_q \delta_q \tilde{\mathbf{h}}^T C \mu_p \delta_p \tilde{\mathbf{h}}^T + \frac{\sigma_6}{8h^6} \sum_{p=1}^{n} \delta_p^2 \tilde{\mathbf{h}} \delta_p^2 \tilde{\mathbf{h}}^T C \delta_p^2 \tilde{\mathbf{h}}^T$$

$$+ \quad \frac{\sigma_4 \sigma_2}{8h^6} \sum_{\substack{p,q=1 \\ p \neq q}}^{n} \Big[\delta_p^2 \tilde{\mathbf{h}} \delta_p^2 \tilde{\mathbf{h}}^T C \delta_q^2 \tilde{\mathbf{h}}^T + \delta_p^2 \tilde{\mathbf{h}} \delta_q^2 \tilde{\mathbf{h}}^T C \delta_p^2 \tilde{\mathbf{h}}^T + \delta_p^2 \tilde{\mathbf{h}} \delta_q^2 \tilde{\mathbf{h}}^T C \delta_q^2 \tilde{\mathbf{h}}^T \Big]$$

$$
\begin{aligned}
&+ \quad \frac{\sigma_4\sigma_2}{8h^6}\sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n}\delta_p^2\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T C\delta_r^2\tilde{\mathbf{h}}^T + \frac{\sigma_4\sigma_2}{8h^6}\sum_{\substack{p,q=1\\p\neq q}}^{n}\delta_p^2\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T C\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\\
&+ \quad \frac{\sigma_4\sigma_2}{8h^6}\sum_{\substack{p,q=1\\p\neq q}}^{n}\delta_q^2\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T C\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T + \frac{\sigma_2^3}{8h^6}\sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n}\delta_p^2\tilde{\mathbf{h}}\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}^T C\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}^T\\
&+ \quad \frac{\sigma_4\sigma_2}{8h^6}\sum_{\substack{p,q=1\\p\neq q}}^{n}\left[\delta_q^2\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T C\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}^T + \mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T C\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\right]\\
&+ \quad \frac{\sigma_4\sigma_2}{8h^6}\sum_{\substack{p,q=1\\p\neq q}}^{n}\left[\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T C\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}^T + \mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T C\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}^T\right.\\
&\quad\left.+\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T C\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\right]\\
&+ \quad \frac{\sigma_2^3}{8h^6}\sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n}\left[\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\delta_r^2\tilde{\mathbf{h}}^T C\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T + \mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\delta_r^2\tilde{\mathbf{h}}^T C\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}^T\right.\\
&\quad\left.+\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T C\delta_r^2\tilde{\mathbf{h}}^T\right]\\
&+ \quad \frac{\sigma_4\sigma_2}{8h^6}\sum_{\substack{p,q=1\\p\neq q}}^{n}\left(\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T C\delta_p^2\tilde{\mathbf{h}}^T + \mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T C\delta_q^2\tilde{\mathbf{h}}^T\right)\\
&+ \quad \frac{\sigma_2^3}{8h^6}\sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n}\left(\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}^T C\mu_p\delta_p\mu_r\delta_r\tilde{\mathbf{h}}^T\right.\\
&\quad\left.+\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}^T C\mu_r\delta_r\mu_p\delta_p\tilde{\mathbf{h}}^T + \mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_r\delta_r\tilde{\mathbf{h}}^T C\mu_r\delta_r\mu_q\delta_q\tilde{\mathbf{h}}^T\right)\\
&+ \quad \frac{\sigma_2^3}{8h^6}\sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n}\left(\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_r\delta_r\tilde{\mathbf{h}}^T C\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}^T\right)\\
&\quad+\frac{\sigma_2}{4h^2}\sum_{p=1}^{n}\left(\delta_p^2\tilde{\mathbf{h}}\sum_{i=1}^{n}R_{i,i} + 2\delta_p^2 R\tilde{\mathbf{h}}\right)\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}\left[\mathcal{A}C^T\mathcal{A}\mathcal{A}^T|Y_{t_{k-1}}\right] &= \mathbb{E}\left[\mathcal{A}\mathcal{A}^T C\mathcal{A}^T|Y_{t_{k-1}}\right]\\
\mathbb{E}\left[\mathcal{A}C^T\mathcal{A}C^T|Y_{t_{k-1}}\right] &= \mathbb{E}\left[\mathcal{A}C^T C\mathcal{A}^T|Y_{t_{k-1}}\right]\\
\mathbb{E}\left[\mathcal{A}C^T C\mathcal{A}^T|Y_{t_{k-1}}\right] &= \sum_{p=1}^{n}\frac{\sigma_2}{h^2}\mu_p\delta_p\tilde{\mathbf{h}}(\hat{z})C^T C\mu_p\delta_p\tilde{\mathbf{h}}^T + \frac{\sigma_4}{4h^4}\delta_p^2\tilde{\mathbf{h}}C^T C\delta_p^2\tilde{\mathbf{h}}^T\\
&+ \quad \frac{\sigma_2^2}{4h^4}\sum_{\substack{p,q=1\\q\neq p}}^{n}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}C^T C\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}} + \delta_p^2\tilde{\mathbf{h}}C^T C\delta_q^2\tilde{\mathbf{h}}^T + RC^T C
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}\left[\mathcal{A}\mathcal{A}^T CC^T | Y_{t_{k-1}}\right] &= \left[\sum_{p=1}^{n} \frac{\sigma_2}{h^2}\mu_p\delta_p\tilde{\mathbf{h}}(\hat{z})\mu_p\delta_p\tilde{\mathbf{h}}^T + \frac{\sigma_4}{4h^4}\left(\delta_p^2\tilde{\mathbf{h}}(\hat{z})\right)\left(\delta_p^2\tilde{\mathbf{h}}(\hat{z})\right)^T\right] CC^T \\
&+ \left[\frac{\sigma_2^2}{4h^4}\sum_{\substack{p,q=1\\ q\neq p}}^{n} \mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}} + \delta_p^2\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T + R\right] CC^T \\
\mathbb{E}\left[\mathcal{A}C^T CC^T | Y_{t_{k-1}}\right] &= CC^T CC^T
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}\left[\mathcal{A}C^T\mathcal{A}\mathcal{A}^T | Y_{t_{k-1}}\right] = S_x \left(\frac{\sigma_4}{2h^4}\sum_{p=1}^{n}\mu_p\delta_p\tilde{\mathbf{h}}C^T\mu_p\delta_p\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\right. \\
+ \quad & \frac{\sigma_2^2}{2h^4}\sum_{\substack{p,q=1\\ p\neq q}}^{n}\mu_p\delta_p\tilde{\mathbf{h}}C^T\mu_p\delta_p\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T + \frac{\sigma_2^2}{2h^4}\sum_{\substack{p,q=1\\ p\neq q}}^{n}\mu_p\delta_p\tilde{\mathbf{h}}C^T\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T \\
+ \quad & \frac{\sigma_4}{2h^4}\sum_{p=1}^{n}\mu_p\delta_p\tilde{\mathbf{h}}C^T\delta_p^2\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T + \frac{\sigma_2^2}{2h^4}\sum_{\substack{p,q=1\\ p\neq q}}^{n}(\mu_p\delta_p\tilde{\mathbf{h}}C^T\delta_q^2\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T) \\
& + \frac{\sigma_2^2}{2h^4}\sum_{\substack{p,q=1\\ p\neq q}}^{n}\left[\mu_p\delta_p\tilde{\mathbf{h}}C^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T + \mu_q\delta_q\tilde{\mathbf{h}}C^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\right] \\
+ \quad & \frac{\sigma_4}{2h^4}\sum_{p=1}^{n}\delta_p^2\tilde{\mathbf{h}}C^T\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T + \frac{\sigma_2^2}{2h^4}\sum_{\substack{p,q=1\\ p\neq q}}^{n}\delta_p^2\tilde{\mathbf{h}}C^T\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T \\
+ \quad & \frac{\sigma_2^2}{2h^4}\sum_{\substack{p,q=1\\ p\neq q}}^{n}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}C^T\mu_p\delta_p\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T + \frac{\sigma_2^2}{2h^4}\sum_{\substack{p,q=1\\ p\neq q}}^{n}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}C^T\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T \\
+ \quad & \frac{\sigma_6}{8h^6}\sum_{p=1}^{n}(\delta_p^2\tilde{\mathbf{h}}C^T\delta_p^2\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T) + \frac{\sigma_4\sigma_2}{8h^6}\sum_{\substack{p,q=1\\ p\neq q}}^{n}\delta_p^2\tilde{\mathbf{h}}C^T\delta_p^2\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T \\
+ \quad & \frac{\sigma_4\sigma_2}{8h^6}\sum_{\substack{p,q=1\\ p\neq q}}^{n}\delta_p^2\tilde{\mathbf{h}}C^T\delta_q^2\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T + \frac{\sigma_4\sigma_2}{8h^6}\sum_{\substack{p,q=1\\ p\neq q}}^{n}\delta_p^2\tilde{\mathbf{h}}C^T\delta_q^2\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T \\
+ \quad & \frac{\sigma_4\sigma_2}{8h^6}\sum_{\substack{p,q,r=1\\ p\neq q\neq r}}^{n}\delta_p^2\tilde{\mathbf{h}}C^T\delta_q^2\tilde{\mathbf{h}}\delta_r^2\tilde{\mathbf{h}}^T + \frac{\sigma_4\sigma_2}{8h^6}\sum_{\substack{p,q=1\\ p\neq q}}^{n}\delta_p^2\tilde{\mathbf{h}}C^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T \\
& + \frac{\sigma_4\sigma_2}{8h^6}\sum_{\substack{p,q=1\\ p\neq q}}^{n}\delta_q^2\tilde{\mathbf{h}}C^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T \\
+ \quad & \frac{\sigma_4\sigma_2}{8h^6}\sum_{\substack{p,q=1\\ p\neq q}}^{n}\delta_q^2\tilde{\mathbf{h}}C^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}^T + \frac{\sigma_2^3}{8h^6}\sum_{\substack{p,q,r=1\\ p\neq q\neq r}}^{n}\delta_p^2\tilde{\mathbf{h}}C^T\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}^T
\end{aligned}
$$

$$+ \quad \frac{\sigma_4\sigma_2}{8h^6}\sum_{\substack{p,q=1\\p\neq q}}^{n}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}C^T\delta_p^2\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T$$

$$+ \quad \frac{\sigma_4\sigma_2}{8h^6}\sum_{\substack{p,q=1\\p\neq q}}^{n}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}C^T\delta_p^2\tilde{\mathbf{h}}\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}^T + \frac{\sigma_4\sigma_2}{8h^6}\sum_{\substack{p,q=1\\p\neq q}}^{n}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}C^T\delta_q^2\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T$$

$$+ \quad \frac{\sigma_2^3}{8h^6}\sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}C^T\delta_r^2\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T + \frac{\sigma_2^3}{8h^6}\sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}C^T\delta_r^2\tilde{\mathbf{h}}\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}^T$$

$$+ \quad \frac{\sigma_2^3}{8h^6}\sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}C^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\delta_r^2\tilde{\mathbf{h}}^T + \frac{\sigma_4\sigma_2}{8h^6}\sum_{\substack{p,q=1\\p\neq q}}^{n}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}C^T\delta_q^2\tilde{\mathbf{h}}\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}^T$$

$$+ \quad \frac{\sigma_4\sigma_2}{8h^6}\sum_{\substack{p,q=1\\p\neq q}}^{n}\left(\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}C^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T + \mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}C^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T\right)$$

$$+ \quad \frac{\sigma_2^3}{8h^6}\sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n}\Big(\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}C^T\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}\mu_p\delta_p\mu_r\delta_r\tilde{\mathbf{h}}^T$$

$$+\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}C^T\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}\mu_r\delta_r\mu_p\delta_p\tilde{\mathbf{h}}^T + \mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}C^T\mu_p\delta_p\mu_r\delta_r\tilde{\mathbf{h}}\mu_r\delta_r\mu_q\delta_q\tilde{\mathbf{h}}^T\Big)$$

$$+ \quad \frac{\sigma_2^3}{8h^6}\sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n}\left(\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}C^T\mu_p\delta_p\mu_r\delta_r\tilde{\mathbf{h}}\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}^T\right)$$

$$+ \quad \frac{\sigma_2}{4h^2}\sum_{p=1}^{n}\left(\delta_p^2 h_i e_j R_{j,j} + \delta_p^2 h_j e_i R_{i,i}\right)\Bigg) + \frac{\sigma_2}{2h^2}\sum_{p=1}^{n}\delta_p^2\tilde{\mathbf{h}}^T\delta_{i,j}R_{i,j}$$

B.3 Proof of Lemma 5.1

Proof.

$$\begin{aligned}
r_{0,2}(t_k|t_{k-1})_{\{\tilde{\mathbf{h}},\tilde{\mathbf{h}}^T\}} &= \mathbb{E}\left[(y(t_k)-\hat{y}(t_k|t_{k-1}))(y(t_k)-\hat{y}(t_k|t_{k-1}))^T|Y_{k-1}\right] \\
&= \mathbb{E}\Bigg[\left(\tilde{D}_{\Delta z}\tilde{\mathbf{h}} + \frac{1}{2}\tilde{D}_{\Delta z}^2\tilde{\mathbf{h}} + v - C\right)\times \\
&\quad \left(\tilde{D}_{\Delta z}\tilde{\mathbf{h}} + \frac{1}{2}\tilde{D}_{\Delta z}^2\tilde{\mathbf{h}} + v - C\right)^T|Y_{k-1}\Bigg] \\
&= \mathbb{E}\left[\left(\tilde{D}_{\Delta z}\tilde{\mathbf{h}} + \frac{1}{2}\tilde{D}_{\Delta z}^2\tilde{\mathbf{h}} + v\right)\left(\tilde{D}_{\Delta z}\tilde{\mathbf{h}} + \frac{1}{2}\tilde{D}_{\Delta z}^2\tilde{\mathbf{h}} + v\right)^T|Y_{k-1}\right] \\
&- \mathbb{E}\left(\tilde{D}_{\Delta z}\tilde{\mathbf{h}} + \frac{1}{2}\tilde{D}_{\Delta z}^2\tilde{\mathbf{h}} + v\right)C^T - C\mathbb{E}\left(\tilde{D}_{\Delta z}\tilde{\mathbf{h}} + \frac{1}{2}\tilde{D}_{\Delta z}^2\tilde{\mathbf{h}} + v\right)^T \\
&\quad +CC^T
\end{aligned}$$

$$
\begin{aligned}
r_{1,1}(t_k|t_{k-1}) &= \mathbb{E}\left[(x(t_k)-\hat{x}(t_k|t_{k-1}))(y(t_k)-\hat{y}(t_k|t_{k-1}))^T|Y_{k-1}\right]\\
&= \mathbb{E}\left[\Delta x(t_k)(y(t_k)-\hat{y}(t_k|t_{k-1}))^T|Y_{k-1}\right]\\
&= \mathbb{E}\left[S_x\Delta z(t_k)\left(\tilde{D}_{\Delta z}\tilde{\mathbf{h}}+\frac{1}{2}\tilde{D}^2_{\Delta z}\tilde{\mathbf{h}}+v-C\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
r_{1,2}(t_k|t_{k-1}) &= \mathbb{E}\left[S_x\Delta z(t_k)(y(t_k)-\hat{y}(t_k|t_{k-1}))^T\left(\mathbb{Y}(t_k)-\hat{\mathbb{Y}}(t_k|t_{k-1})\right)^T\right]\\
&\quad +\mathbb{E}\left[\hat{x}(t_k|t_{k-1})(y(t_k)-\hat{y}(t_k|t_{k-1}))^T\left(\mathbb{Y}(t_k)-\hat{\mathbb{Y}}(t_k|t_{k-1})\right)^T\right]\\
&= \mathbb{E}\left[S_x\Delta z(t_k)\left\{(y_1(t_k)-\hat{y}_1(t_k|t_{k-1}))(y(t_k)-\hat{y}(t_k|t_{k-1}))^T,\ldots,\right.\right.\\
&\quad \left.\left.(y_n(t_k)-\hat{y}_n(t_k|t_{k-1}))(y(t_k)-\hat{y}(t_k|t_{k-1}))^T\right\}\right]\\
&+ \hat{x}(t_k|t_{k-1})\mathbb{E}\left[(y_1(t_k)-\hat{y}_1(t_k|t_{k-1}))(y(t_k)-\hat{y}(t_k|t_{k-1}))^T,\ldots,\right.\\
&\quad \left.(y_n(t_k)-\hat{y}_n(t_k|t_{k-1}))(y(t_k)-\hat{y}(t_k|t_{k-1}))^T\right]
\end{aligned}
$$

$$
r_{0,3}(t_k|t_{k-1}) = \left(\mathbb{E}\left(\Delta y_i(t_k)\Delta y_j(t_k)\Delta y(t_k)^T\right)\right)_{1\le i\le n,1\le j\le n}
$$

$$
\begin{aligned}
r_{0,4}(t_k|t_{k-1}) &= \mathbb{E}\left[\left(\tilde{D}_{\Delta z}\tilde{\mathbf{h}}+\frac{1}{2}\tilde{D}^2_{\Delta z}\tilde{\mathbf{h}}+v-C\right)\left(\tilde{D}_{\Delta z}\tilde{\mathbf{h}}+\frac{1}{2}\tilde{D}^2_{\Delta z}\tilde{\mathbf{h}}+v-C\right)^T\right.\\
&\quad \left.\left(\tilde{D}_{\Delta z}\tilde{\mathbf{h}}+\frac{1}{2}\tilde{D}^2_{\Delta z}\tilde{\mathbf{h}}+v-C\right)\left(\tilde{D}_{\Delta z}\tilde{\mathbf{h}}+\frac{1}{2}\tilde{D}^2_{\Delta z}\tilde{\mathbf{h}}+v-C\right)^T|Y_{k-1}\right]\\
&= \mathbb{E}\left[\mathcal{A}\mathcal{A}^T\mathcal{A}\mathcal{A}^T|Y_{t_{k-1}}\right]-\mathbb{E}\left[\mathcal{A}\mathcal{A}^T\mathcal{A}C^T|Y_{t_{k-1}}\right]-\mathbb{E}\left[\mathcal{A}\mathcal{A}^TC\mathcal{A}^T|Y_{t_{k-1}}\right]\\
&+ \mathbb{E}\left[\mathcal{A}\mathcal{A}^TCC^T\right]-\mathbb{E}\left[\mathcal{A}C^T\mathcal{A}\mathcal{A}^T\right]+\mathbb{E}\left[\mathcal{A}C^T\mathcal{A}C^T\right]\\
&+ \mathbb{E}\left[\mathcal{A}C^TC\mathcal{A}^T\right]-\mathbb{E}\left[\mathcal{A}C^TCC^T\right]
\end{aligned}
$$

$$
\begin{aligned}
M_{0,2}(t_k|t_{k-1}) &= \left(\mathbb{E}\left(\Delta y(t_k)\Delta y_i(t_k)\Delta y_j(t_k)\Delta y(t_k)^T\right)\right)_{1\le i\le n,1\le j\le n}\\
r_{2,2}(t_k|t_{k-1}) &= \mathbb{E}\left[\Delta x(t_k)\Delta x(t_k)^T\Delta y(t_k)\Delta y(t_k)^T\right]\\
&= \mathbb{E}\left[S_x\sum_{k=1}^{n}\left[\Delta z_i\Delta z_k S_{j,k}\right]\left(\tilde{D}_{\Delta z}\tilde{\mathbf{h}}+\frac{1}{2}\tilde{D}^2_{\Delta z}\tilde{\mathbf{h}}+v-C\right)\right.\\
&\quad \left.\left(\tilde{D}_{\Delta z}\tilde{\mathbf{h}}+\frac{1}{2}\tilde{D}^2_{\Delta z}\tilde{\mathbf{h}}+v-C\right)^T|Y_{k-1}\right]\\
&= S_xQ_{i,j},\quad \text{where } Q_{i,j} \text{ is defined below}\\
r_{1,3}(t_k|t_{k-1}) &= \mathbb{E}\left[\Delta x(t_k)\Delta y(t_k)^T\Delta y(t_k)\Delta y(t_k)^T\right]\\
Q_{i,j} &= \frac{1}{4h^4}\left[\sigma_6S_{j,i}\delta_i^2\tilde{\mathbf{h}}\delta_i^2\tilde{\mathbf{h}}^2+\sum_{p=1}^{n}\sigma_2\sigma_4S_{j,i}\delta_p^2\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\right.
\end{aligned}
$$

$$
\begin{aligned}
& +\sum_{p=1}^{n}\sigma_2\sigma_4 S_{j,i}\left[\delta_i^2\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T+\delta_p^2\tilde{\mathbf{h}}\delta_i^2\tilde{\mathbf{h}}^T\right]+\sum_{\substack{p,q=1\\p\neq q}}^{n}\sigma_2^3 S_{j,i}\delta_p^2\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T\Bigg]\\
+\ & \frac{\sigma_2\sigma_4}{4h^4}\sum_{p=1}^{n}S_{j,p}\left[\delta_i^2\tilde{\mathbf{h}}\mu_i\delta_i\mu_p\delta_p\tilde{\mathbf{h}}^T+\delta_p^2\tilde{\mathbf{h}}\mu_i\delta_i\mu_p\delta_p\tilde{\mathbf{h}}^T+\delta_p^2\tilde{\mathbf{h}}\mu_p\delta_p\mu_i\delta_i\tilde{\mathbf{h}}^T\right.\\
& \left.+\delta_i^2\tilde{\mathbf{h}}\mu_i\delta_i\mu_p\delta_p\tilde{\mathbf{h}}^T\right]\\
+\ & \frac{\sigma_2^3}{4h^4}\sum_{\substack{p,r=1\\p\neq r}}^{n}S_{j,p}\left[\delta_r^2\tilde{\mathbf{h}}\mu_i\delta_i\mu_p\delta_p\tilde{\mathbf{h}}^T+\delta_r^2\tilde{\mathbf{h}}\mu_p\delta_p\mu_i\delta_i\tilde{\mathbf{h}}^T+\mu_i\delta_i\mu_p\delta_p\tilde{\mathbf{h}}\delta_r^2\tilde{\mathbf{h}}^T\right.\\
& \left.+\mu_p\delta_p\mu_i\delta_i\tilde{\mathbf{h}}\delta_r^2\tilde{\mathbf{h}}^T\right]\\
+\ & \frac{\sigma_2\sigma_4}{4h^4}\sum_{p=1}^{n}S_{j,p}\left[\mu_i\delta_i\mu_p\delta_p\tilde{\mathbf{h}}\delta_i^2\tilde{\mathbf{h}}^T+\mu_i\delta_i\mu_p\delta_p\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T+\mu_p\delta_p\mu_i\delta_i\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\right.\\
& \left.+\mu_p\delta_p\mu_i\delta_i\tilde{\mathbf{h}}\delta_i^2\tilde{\mathbf{h}}^T\right]\\
+\ & \frac{\sigma_2^3}{4h^4}\sum_{\substack{p,q=1\\p\neq q}}^{n}S_{j,p}\left[\mu_i\delta_i\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}^T+\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_i\delta_i\tilde{\mathbf{h}}^T\right.\\
& \left.+\mu_i\delta_i\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T+\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_i\delta_i\mu_q\delta_q\tilde{\mathbf{h}}^T\right]\\
+\ & \frac{\sigma_2^3}{4h^4}\sum_{\substack{p,q=1\\p\neq q}}^{n}S_{j,q}\left[\mu_p\delta_p\mu_i\delta_i\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T+\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_i\delta_i\tilde{\mathbf{h}}^T\right.\\
& \left.+\mu_p\delta_p\mu_i\delta_i\tilde{\mathbf{h}}\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}^T+\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_i\delta_i\mu_p\delta_p\tilde{\mathbf{h}}^T\right]\\
+\ & \frac{\sigma_2^3}{4h^4}\sum_{\substack{p,q=1\\p\neq q}}^{n}S_{j,i}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T+\frac{\sigma_2^2}{h^2}\left[\sum_{p=1}^{n}S_{j,i}\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\right.\\
& \left.+S_{j,p}\mu_i\delta_i\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T+S_{j,p}\mu_p\delta_p\tilde{\mathbf{h}}\mu_i\delta_i\tilde{\mathbf{h}}^T\right]\\
+\ & \frac{\sigma_2^2}{2h^2}\sum_{p=1}^{n}S_{j,i}\left[\delta_p^2\tilde{\mathbf{h}}C^T+C\delta_p^2\tilde{\mathbf{h}}^T\right]+\frac{\sigma_2^2}{2h^2}\sum_{p=1}^{n}S_{j,p}\left[\mu_i\delta_i\mu_p\delta_p\tilde{\mathbf{h}}+\mu_p\delta_p\mu_i\delta_i\tilde{\mathbf{h}}\right]C^T\\
+\ & \frac{\sigma_2^2}{2h^2}\sum_{p=1}^{n}S_{j,p}\left[C\mu_i\delta_i\mu_p\delta_p\tilde{\mathbf{h}}^T+C\mu_p\delta_p\mu_i\delta_i\tilde{\mathbf{h}}^T\right]
\end{aligned}
$$

where $\Delta x(t_k) = x(t_k) - \hat{x}(t_k|t_{k-1})$, $\Delta y(t_k) = y(t_k) - \hat{y}(t_k|t_{k-1})$ and $M_{0,2}(t_k|t_{k-1})$ can be generated from $r_{0,4}(t_k|t_{k-1})$ $\square$

B.4 Proof of Lemma 5.2

Proof. From (5.10) and applying Baye's rule, we have

$$P(x(t_k), y(t_k)|Y_{t_{k-1}}) = P(x(t_k)|Y_{t_k})P(y(t_k)|Y_{t_{k-1}}). \tag{B.1}$$

Multiplying equation $(B.1)$ by the product of two arbitrary functions $s(x(t_k))$ and $u(y(t_k))$, and taking the expectations, we have

$$\begin{aligned}
\int\int s(x)u(y)P(x,y|Y_{t_{k-1}})dxdy &= \int\int s(x)u(y)P(x|Y_{t_k})P(y|Y_{t_{k-1}})dxdy \\
&= \int\left[\int s(x)P(x|Y_{t_k})dx\right]u(y)P(y|Y_{t_{k-1}})dy \\
&= \mathbb{E}\left[\mathbb{E}\left[s(x(t_k))|Y_{t_k}\right]u(y(t_k))Y_{t_{k-1}}\right].
\end{aligned}$$

Hence,

$$\mathbb{E}\left[s(x(t_k))u(y(t_k))|Y_{t_{k-1}}\right] = \mathbb{E}\left[\mathbb{E}\left[s(x(t_k))|Y_{t_k}\right]u(y(t_k))|Y_{t_{k-1}}\right]. \tag{B.2}$$

Equation $(B.2)$ provides a systematic feasible procedure for solving for A_i, B_i, $i = 0, 1$, and A_2. Substituting $s = x(t_k)$ and $u = 1$, we have

$$\mathbb{E}\left[x(t_k)|Y_{t_{k-1}}\right] = \mathbb{E}\left[\mathbb{E}\left[x(t_k)|Y_{t_k}\right]|Y_{t_{k-1}}\right]. \tag{B.3}$$

Hence

$$\mathbb{E}\left[x(t_k)|Y_{t_{k-1}}\right] = \mathbb{E}\left[\mathbb{E}\left[x(t_k)|Y_{t_k}\right]|Y_{t_{k-1}}\right].$$

This implies that

$$\begin{aligned}
\hat{x}(t_k|t_{k-1}) &= \mathbb{E}\left[\hat{x}(t_k|t_k)|Y_{t_{k-1}}\right] \\
&= \mathbb{E}\left[A_0 + A_1(y(t_k) - \hat{y}(t_k|t_{k-1})) + A_2(\mathbb{Y} - \hat{\mathbb{Y}})(y(t_k) - \hat{y}(t_k|t_{k-1}))|Y_{t_{k-1}}\right].
\end{aligned}$$

Thus,

$$r_{1,0}(t_k|t_{k-1}) = A_0(t_k|t_{k-1}) + A_2(t_k|t_{k-1})\sigma_{Y2}(t_k|t_{k-1}), \tag{B.4}$$

where $r_{1,0}(t_k|t_{k-1}) = \hat{x}(t_k|t_{k-1})$. Substituting $s = x(t_k)$ and $u = (y(t_k) - \hat{y}(t_k|t_{k-1}))^T$, we have

$$\begin{aligned}
&\mathbb{E}\left[x(t_k)(y(t_k) - \hat{y}(t_k|t_{k-1}))^T|Y_{t_{k-1}}\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[x(t_k)|Y_{t_k}\right](y(t_k) - \hat{y}(t_k|t_{k-1}))^T|Y_{t_{k-1}}\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[(A_0 + A_1(y(t_k) - \hat{y}(t_k|t_{k-1}))\right.\right. \\
&\left.\left.\left. +A_2(\mathbb{Y} - \hat{\mathbb{Y}})(y(t_k) - \hat{y}(t_k|t_{k-1}))\right)(y(t_k) - \hat{y}(t_k|t_{k-1}))^T\right]|Y_{t_{k-1}}\right],
\end{aligned}$$

Hence,

$$r_{1,1}(t_k|t_{k-1}) = A_1(t_k|t_{k-1})r_{0,2}(t_k|t_{k-1}) + A_2(t_k|t_{k-1})r_{0,3}^T(t_k|t_{k-1}). \tag{B.5}$$

Lastly, substituting $s = x(t_k)$ and $u = (y(t_k) - \hat{y}(t_k|t_{k-1}))^T (\mathbb{Y} - \hat{\mathbb{Y}})^T$, we have

$$\begin{aligned}
&\mathbb{E}\left[x(t_k)(y(t_k) - \hat{y}(t_k|t_{k-1}))^T(\mathbb{Y} - \hat{\mathbb{Y}})^T|Y_{t_{k-1}}\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[x(t_k)|Y_{t_k}\right](y(t_k) - \hat{y}(t_k|t_{k-1}))^T(\mathbb{Y} - \hat{\mathbb{Y}})^T|Y_{t_{k-1}}\right].
\end{aligned} \tag{B.6}$$

Hence,

$$\begin{aligned}
r_{1,2}(t_k|t_{k-1}) &= A_0(t_k|t_{k-1})\sigma_{Y2}^T(t_k|t_{k-1}) + A_1(t_k|t_{k-1})r_{0,3}(t_k|t_{k-1}) \\
&\quad +A_2(t_k|t_{k-1})M_{0,2}(t_k|t_{k-1}).
\end{aligned} \tag{B.7}$$

The result follows by solving the systems of linear equations $(B.4)$, $(B.5)$, $(B.7)$. □

B.5 Proof of Lemma 5.3

Proof.

First, we substitute $s = (x(t_k) - \hat{x}(t_k|t_k))(x(t_k) - \hat{x}(t_k|t_k))^T$ and $u = 1$ into equation $(B.2)$ and obtain

$$\begin{aligned}
&\mathbb{E}\left[(x(t_k) - \hat{x}(t_k|t_k))(x(t_k) - \hat{x}(t_k|t_k))^T|Y_{t_{k-1}}\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[(x(t_k) - \hat{x}(t_k|t_k))(x(t_k) - \hat{x}(t_k|t_k))^T|Y_{t_k}\right]|Y_{t_{k-1}}\right] \\
&= \mathbb{E}\left[P(t_k|t_k)\right].
\end{aligned}$$

Hence,

$$N_1 = B_0 + B_1 r_{0,2}(t_k|t_{k-1}). \tag{B.8}$$

Lastly, substituting

$$\begin{aligned}
s &= (x(t_k) - \hat{x}(t_k|t_k))(x(t_k) - \hat{x}(t_k|t_k))^T \\
u &= (y(t_k) - \hat{y}(t_k|t_{k-1}))(y(t_k) - \hat{y}(t_k|t_{k-1}))^T
\end{aligned}$$

into equation $(B.2)$

$$N_2 = B_0 r_{0,2}(t_k|t_{k-1}) + B_1 r_{0,4}. \tag{B.9}$$

The fifth and upper moments of $y(t_k) - \hat{y}(t_k|t_{k-1})$ is neglected in N_2.

$$\mathbb{E}\left[(x(t_k) - \hat{x}(t_k|t_{k-1}))(y(t_k) - \hat{y}(t_k|t_{k-1}))^T A_1^T (y(t_k) - \hat{y}(t_k|t_{k-1}))\times\right.$$
$$\left.(y(t_k) - \hat{y}(t_k|t_{k-1}))^T|Y_{k-1}\right]$$

can be generated from $r_{1,3}$,

$$\mathbb{E}\left[(x(t_k|t_{k-1}) - A_0)(y(t_k) - \hat{y}(t_k|t_{k-1}))^T A_1^T (y(t_k) - \hat{y}(t_k|t_{k-1}))\times\right.$$
$$\left.(y(t_k) - \hat{y}(t_k|t_{k-1}))^T|Y_{t_{k-1}}\right]$$

can be generated from $r_{0,3}$,

$$\mathbb{E}\left[A_1(y(t_k) - \hat{y}(t_k|t_{k-1}))(x(t_k) - \hat{x}(t_k|t_{k-1}))^T (y(t_k) - \hat{y}(t_k|t_{k-1}))\times\right.$$
$$\left.(y(t_k) - \hat{y}(t_k|t_{k-1}))^T|Y_{t_{k-1}}\right]$$

can be generated from $r_{1,3}$,

$$\mathbb{E}\left[A_1(y(t_k) - \hat{y}(t_k|t_{k-1}))(x(t_k|t_{k-1}) - A_0)^T (y(t_k) - \hat{y}(t_k|t_{k-1}))\times\right.$$
$$\left.(y(t_k) - \hat{y}(t_k|t_{k-1}))^T|Y_{t_{k-1}}\right]$$

can be generated from $r_{0,3}$,

$$\mathbb{E}\left[A_1(y(t_k) - \hat{y}(t_k|t_{k-1}))(y(t_k) - \hat{y}(t_k|t_{k-1}))^T A_1^T (y(t_k) - \hat{y}(t_k|t_{k-1}))\times\right.$$
$$\left.(y(t_k) - \hat{y}(t_k|t_{k-1}))^T|Y_{t_{k-1}}\right]$$

can be generated from $r_{0,4}$, where $r_{1,3}$, $r_{0,3}$, and $r_{0,4}$ are defined in Appendix $B.3$. The conclusion of the Lemma follows by solving the systems of equation $(B.8)$ and $(B.9)$. $\square$

Appendix C

C.1 Proof of Lemma 6.1

Proof of Lemma 6.1 for small m_k, $m_{k-1} \leq m_k$, *Proof.*

$$\begin{aligned}
\bar{S}_{m_k,k} &= \frac{1}{m_k}\sum_{i=1-m_k}^{0} F^i x_{k-1} = \frac{1}{m_k}\left[\sum_{i=1-m_k}^{-1-m_{k-1}} F^i x_{k-1} + \sum_{i=-m_{k-1}}^{-1} F^i x_{k-1} + F^0 x_{k-1}\right] \\
&= \frac{1}{m_k}\left[m_{k-1}\bar{S}_{m_{k-1},k-1} + \sum_{i=1-m_k}^{1-m_{k-1}} F^i x_{k-1} - F^{1-m_{k-1}} x_{k-1} - F^{-m_{k-1}} x_{k-1}\right. \\
&\quad \left. + F^0 x_{k-1}\right]
\end{aligned}$$

$$\begin{aligned}
s^2_{m_k,k} &= \frac{1}{m_k}\left[\sum_{i=-m_k+1}^{0}\left(F^i x_{k-1}\right)^2 - \frac{1}{m_k}\left(\sum_{j=-m_k+1}^{0} F^j x_{k-1}\right)^2\right] \\
&= \frac{1}{m_k}\left[\sum_{i=-m_k+1}^{-m_{k-1}-1}\left(F^i x_{k-1}\right)^2 + \sum_{i=-m_{k-1}}^{-1}\left(F^i x_{k-1}\right)^2 + \left(F^0 x_{k-1}\right)^2\right. \\
&\quad \left. - \frac{1}{m_k}\left(\sum_{j=-m_k+1}^{0} F^j x_{k-1}\right)^2\right] \\
&= \frac{1}{m_k}\left[\sum_{i=-m_{k-1}}^{-1}\left(F^i x_{k-1}\right)^2 - \frac{1}{m_{k-1}}\left(\sum_{i=-m_{k-1}}^{-1} F^i x_{k-1}\right)^2 + \right. \\
&\quad \left. \frac{1}{m_{k-1}}\left(\sum_{i=-m_{k-1}}^{-1} F^i x_{k-1}\right)^2\right] \\
&\quad + \frac{1}{m_k}\left[\left(F^0 x_{k-1}\right)^2 - \left(F^{-m_{k-1}} x_{k-1}\right)^2 - \left(F^{-m_{k-1}+1} x_{k-1}\right)^2\right. \\
&\quad \left. - \frac{1}{m_k}\left(\sum_{i=-m_k+1}^{0} F^i x_{k-1}\right)^2 + \sum_{i=-m_k+1}^{-m_{k-1}+1}\left(F^i x_{k-1}\right)^2\right]
\end{aligned}$$

$$= \frac{m_{k-1}}{m_k} s^2_{m_{k-1},k-1} + \frac{m_{k-1}}{m_k} \bar{S}^2_{m_{k-1},k-1} - \bar{S}^2_{m_k,k} + \frac{\left(F^0 x_{k-1}\right)^2 - \left(F^{-m_{k-1}} x_{k-1}\right)^2 - \left(F^{-m_{k-1}+1} x_{k-1}\right)^2}{m_k} + \frac{\sum_{i=-m_k+1}^{-m_{k-1}+1} \left(F^i x_{k-1}\right)^2}{m_k}.$$

Hence,

$$s^2_{m,k} = \frac{m_{k-1}}{m_k} s^2_{m_{k-1},k-1} + \frac{m_{k-1}}{m_k} \bar{S}^2_{m_{k-1},k-1} - \bar{S}^2_{m_k,k} + \frac{\left(F^0 x_{k-1}\right)^2 - \left(F^{-m_{k-1}} x_{k-1}\right)^2 - \left(F^{-m_{k-1}+1} x_{k-1}\right)^2}{m_k} + \frac{\sum_{i=-m_k+1}^{-m_{k-1}+1} \left(F^i x_{k-1}\right)^2}{m_k}. \quad \text{(C.1)}$$

Next, we find an expression connecting $\bar{S}^2_{m_k,k}$, $\bar{S}^2_{m_k-1,k-1}$ and $s^2_{m_k-1,k-1}$. By definition and simplification,

$$\begin{aligned} m_k^2 \bar{S}^2_{m_k,k} &= \left[\sum_{i=-m_k+1}^{0} F^i x_{k-1}\right]^2 = \sum_{i=-m_k+1}^{0} \left(F^i x_{k-1}\right)^2 + \sum_{\substack{l,s=-m_k+1\\ l\neq s}}^{0} F^l x_{k-1} F^s x_{k-1} \\ &= (m_{k-1}) s^2_{m_{k-1},k-1} + m_{k-1} \bar{S}^2_{m_{k-1},k-1} + (F^0 x_{k-1})^2 - (F^{-m_{k-1}} x_{k-1})^2 \\ &\quad -(F^{-m_{k-1}+1} x_{k-1})^2 + \sum_{i=-m_k+1}^{-m_{k-1}+1} (F^i x_{k-1})^2 + \sum_{\substack{l,s=-m_k+1\\ l\neq s}}^{0} F^l x_{k-1} F^s x_{k-1} \end{aligned} \quad \text{(C.2)}$$

Substituting $(C.2)$ into $(C.1)$, we have

$$s^2_{m,k} = \frac{m_k-1}{m_k} \left[\frac{m_{k-1}}{m_k} s^2_{m_{k-1},k-1} + \frac{m_{k-1}}{m_k} \bar{S}^2_{m_{k-1},k-1} + \frac{\left(F^0 x_{k-1}\right)^2 - \left(F^{-m_{k-1}} x_{k-1}\right)^2 - \left(F^{-m_{k-1}+1} x_{k-1}\right)^2}{m_k} + \frac{\sum_{i=-m_k+1}^{-m_{k-1}+1} \left(F^i x_{k-1}\right)^2}{m_k} \right] - \frac{\sum_{\substack{l,s=-m_k+1\\ l\neq s}}^{0} F^l x_{k-1} F^s x_{k-1}}{m_k(m_k-1)}. \quad \text{(C.3)}$$

Likewise, using equation $(C.2)$,

$$\begin{aligned} m^2_{k-1} \bar{S}^2_{m_{k-1},k-1} &= (m_{k-2}) s^2_{m_{k-2},k-2} + m_{k-2} \bar{S}^2_{m_{k-2},k-2} + (F^{-1} x_{k-1})^2 - (F^{-m_{k-2}-1} x_{k-1})^2 \\ &\quad -(F^{-m_{k-2}} x_{k-1})^2 + \sum_{i=-m_{k-1}}^{-m_{k-2}} (F^i x_{k-1})^2 + \sum_{\substack{l,s=-m_{k-1}\\ l\neq s}}^{-1} F^l x_{k-1} F^s x_{k-1}. \end{aligned}$$

Also,

$$\begin{aligned} m^2_{k-2} \bar{S}^2_{m_{k-2},k-2} &= (m_{k-3}) s^2_{m_{k-3},k-3} + m_{k-3} \bar{S}^2_{m_{k-3},k-3} + (F^{-2} x_{k-1})^2 - (F^{-m_{k-3}-2} x_{k-1})^2 \\ &\quad -(F^{-m_{k-3}-1} x_{k-1})^2 + \sum_{i=-m_{k-2}-1}^{-m_{k-3}-1} (F^i x_{k-1})^2 + \sum_{\substack{l,s=-m_{k-2}-1\\ l\neq s}}^{-2} F^l x_{k-1} F^s x_{k-1}. \end{aligned}$$

Continuing in this sense and substituting $\bar{S}^2_{m_{k-i},k-i}$, $i = 2, ..., p-1$ into $\bar{S}^2_{m_{k-1},k-1}$, we have

$$
\begin{aligned}
(m_{k-1})\bar{S}^2_{m_k-1,k-1} \quad = \quad & \sum_{i=2}^{p}\left[\frac{m_{k-i}}{\prod_{j=1}^{i-1} m_{k-j}}\right] s^2_{m_{k-i},k-i} + \frac{m_{k-p}}{\prod_{j=1}^{p-1} m_{k-j}}\bar{S}^2_{m_{k-p},k-p} + \sum_{i=2}^{p}\frac{\left(F^{-i+1}x_{k-1}\right)^2}{\prod_{j=1}^{i-1} m_{k-j}} \\
& - \sum_{i=2}^{p}\frac{\left(F^{-i+1-m_{k-i}}x_{k-1}\right)^2}{\prod_{j=1}^{i-1} m_{k-j}} - \sum_{i=2}^{p}\frac{\left(F^{-i+2-m_{k-i}}x_{k-1}\right)^2}{\prod_{j=1}^{i-1} m_{k-j}} \\
& + \sum_{i=2}^{p}\left[\frac{\sum_{l=-i+2-m_{k-i+1}}^{-i+2-m_{k-i}} (F^l x_{k-1})^2}{\prod_{j=1}^{i-1} m_{k-j}}\right] + \sum_{i=2}^{p}\left[\frac{\sum_{\substack{l,s=-i+2-m_{k-i+1} \\ l\neq s}}^{-i+1} F^l x_{k-1}F^s x_{k-1}}{\prod_{j=1}^{i-1} m_{k-j}}\right]
\end{aligned}
\tag{C.4}
$$

Finally, the result follows by substituting $(C.4)$ into $(C.3)$.

□

C.2 Proof Lemma 6.1

Proof of Lemma 6.1 for small m_k, $m_k \leq m_{k-1}$, *Proof.* Following the same steps, if $m_k \leq m_{k-1}$,

$$
\begin{cases}
s^2_{m_k,k} = \frac{m_k-1}{m_k}\left[\sum_{i=1}^{p}\left[\frac{m_{k-i}}{\prod_{j=0}^{i-1} m_{k-j}}\right] s^2_{m_{k-i},k-i} + \frac{m_{k-p}}{\prod_{j=0}^{p-1} m_{k-j}}\bar{S}^2_{m_{k-p},k-p}\right] \\
\qquad + \varpi_{m_{k-1},k-1}, \; m_k \leq m_{k-1} \\
\varpi_{m_{k-1},k-1} = \frac{m_k-1}{m_k}\left[\sum_{i=1}^{p}\frac{\left(F^{-i+1}x_{k-1}\right)^2}{\prod_{j=0}^{i-1} m_{k-j}} - \sum_{i=1}^{p}\left[\frac{\sum_{l=-i+1-m_{k-i}}^{-i+1-m_{k-i+1}} (F^l x_{k-1})^2}{\prod_{j=0}^{i-1} m_{k-j}}\right]\right. \\
\qquad \left. + \sum_{i=1}^{p}\left[\frac{\sum_{\substack{l,s=-i+2-m_{k-i+1} \\ l\neq s}}^{-i+1} F^l x_{k-1}F^s x_{k-1}}{\prod_{j=0}^{i-1} m_{k-j}}\right]\right] - \frac{1}{m_k}\sum_{\substack{l,s=-m_k+1 \\ l\neq s}}^{0} F^l x_{k-1}F^s x_{k-1},
\end{cases}
$$

□

C.3 Proof Lemma 6.1

Proof of Lemma 6.1 for large m_k *Proof.*

$$
\begin{aligned}
s^2_{m_k,k} &= \frac{1}{m_k-1}\left[\sum_{i=-m_k+1}^{0}\left(F^i x_{k-1}\right)^2 - \frac{1}{m_k}\left(\sum_{j=-m_k+1}^{0} F^j x_{k-1}\right)^2\right] \\
&= \frac{1}{m_k-1}\left[\sum_{i=-m_{k-1}}^{-1}\left(F^i x_{k-1}\right)^2 - \frac{1}{m_{k-1}}\left(\sum_{i=-m_{k-1}}^{-1} F^i x_{k-1}\right)^2\right. \\
&\quad \left. + \frac{1}{m_{k-1}}\left(\sum_{i=-m_{k-1}}^{-1} F^i x_{k-1}\right)^2\right] \\
&\quad + \frac{1}{m_k-1}\left[\left(F^0 x_{k-1}\right)^2 - \left(F^{-m_{k-1}} x_{k-1}\right)^2 - \left(F^{-m_{k-1}+1} x_{k-1}\right)^2\right. \\
&\quad \left. - \frac{1}{m_k}\left(\sum_{i=-m_k+1}^{0} F^i x_{k-1}\right)^2 + \left(\sum_{i=-m_k+1}^{-m_{k-1}+1} F^i x_{k-1}\right)^2\right] \\
&= \frac{m_{k-1}-1}{m_k-1} s^2_{m_{k-1},k-1} + \frac{m_{k-1}}{m_k-1}\bar{S}^2_{m_{k-1},k-1} - \frac{m_k}{m_k-1}\bar{S}^2_{m_k,k} \\
&\quad + \frac{\left(F^0 x_{k-1}\right)^2 - \left(F^{-m_{k-1}} x_{k-1}\right)^2 - \left(F^{-m_{k-1}+1} x_{k-1}\right)^2}{m_k-1} + \frac{\sum\limits_{i=-m_k+1}^{-m_{k-1}+1}\left(F^i x_{k-1}\right)^2}{m_k-1}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
s^2_{m,k} &= \tfrac{m_{k-1}-1}{m_k-1} s^2_{m_{k-1},k-1} + \tfrac{m_{k-1}}{m_k-1}\bar{S}^2_{m_{k-1},k-1} - \tfrac{m_k}{m_k-1}\bar{S}^2_{m_k,k} \\
&\quad + \tfrac{\left(F^0 x_{k-1}\right)^2 - \left(F^{-m_{k-1}} x_{k-1}\right)^2 - \left(F^{-m_{k-1}+1} x_{k-1}\right)^2}{m_k-1} + \tfrac{\sum_{i=-m_k+1}^{-m_{k-1}+1}\left(F^i x_{k-1}\right)^2}{m_k-1}.
\end{aligned}
\tag{C.5}
$$

Next, we find an expression connecting $\bar{S}^2_{m_k,k}$, $\bar{S}^2_{m_k-1,k-1}$ and $s^2_{m_k-1,k-1}$. By definition,

$$
\begin{aligned}
m_k^2 \bar{S}^2_{m_k,k} &= \left[\sum_{i=-m_k+1}^{0} F^i x_{k-1}\right]^2 = \sum_{i=-m_k+1}^{0}\left(F^i x_{k-1}\right)^2 + \sum_{\substack{l,s=-m_k+1 \\ l\neq s}}^{0} F^l x_{k-1} F^s x_{k-1} \\
&= (m_{k-1}-1)s^2_{m_{k-1},k-1} + m_{k-1}\bar{S}^2_{m_{k-1},k-1} + (F^0 x_{k-1})^2 - (F^{-m_{k-1}} x_{k-1})^2 \\
&\quad - (F^{-m_{k-1}+1} x_{k-1})^2 + \sum_{i=-m_k+1}^{-m_{k-1}+1}(F^i x_{k-1})^2 + \sum_{\substack{l,s=-m_k+1 \\ l\neq s}}^{0} F^l x_{k-1} F^s x_{k-1}
\end{aligned}
\tag{C.6}
$$

Substituting ($C.6$) into ($C.5$), we have

$$
\begin{aligned}
s^2_{m,k} &= \tfrac{m_{k-1}-1}{m_k} s^2_{m_{k-1},k-1} + \tfrac{m_{k-1}}{m_k}\bar{S}^2_{m_{k-1},k-1} + \tfrac{\left(F^0 x_{k-1}\right)^2 - \left(F^{-m_{k-1}} x_{k-1}\right)^2 - \left(F^{-m_{k-1}+1} x_{k-1}\right)^2}{m_k} \\
&\quad + \tfrac{\sum_{i=-m_k+1}^{-m_{k-1}+1}\left(F^i x_{k-1}\right)^2}{m_k} - \tfrac{\sum_{\substack{l,s=-m_k+1 \\ l\neq s}}^{0} F^l x_{k-1} F^s x_{k-1}}{m_k(m_k-1)}.
\end{aligned}
\tag{C.7}
$$

Likewise,

$$
\begin{aligned}
m_{k-1}^2 \bar{S}_{m_{k-1},k-1}^2 &= (m_{k-2}-1)s_{m_{k-2},k-2}^2 + m_{k-2}\bar{S}_{m_{k-2},k-2}^2 + (F^{-1}x_{k-1})^2 \\
&\quad -(F^{-m_{k-2}-1}x_{k-1})^2 - (F^{-m_{k-2}}x_{k-1})^2 + \sum_{i=-m_{k-1}}^{-m_{k-2}} (F^i x_{k-1})^2 \\
&\quad + \sum_{\substack{l,s=-m_{k-1} \\ l\neq s}}^{-1} F^l x_{k-1} F^s x_{k-1},
\end{aligned}
$$

$$
\begin{aligned}
m_{k-2}^2 \bar{S}_{m_{k-2},k-2}^2 &= (m_{k-3}-1)s_{m_{k-3},k-3}^2 + m_{k-3}\bar{S}_{m_{k-3},k-3}^2 + (F^{-2}x_{k-1})^2 \\
&\quad -(F^{-m_{k-3}-2}x_{k-1})^2 - (F^{-m_{k-3}-1}x_{k-1})^2 + \sum_{i=-m_{k-2}-1}^{-m_{k-3}-1} (F^i x_{k-1})^2 \\
&\quad + \sum_{\substack{l,s=-m_{k-2} \\ l\neq s}}^{-2} F^l x_{k-1} F^s x_{k-1}.
\end{aligned}
$$

Continuing in this sense and substituting $\bar{S}_{m_{k-i},k-i}$, $i = 2, ..., p-1$ into $\bar{S}_{m_{k-1},k-1}$, we have

$$
\begin{aligned}
(m_{k-1})\bar{S}_{m_{k-1},k-1}^2 &= \sum_{i=2}^{p} \left[\frac{m_{k-i}-1}{\prod_{j=1}^{i-1} m_{k-j}} \right] s_{m_{k-i},k-i}^2 + \frac{m_{k-p}}{\prod_{j=1}^{p-1} m_{k-j}} \bar{S}_{m_{k-p},k-p}^2 + \sum_{i=2}^{p} \frac{\left(F^{-i+1}x_{k-1}\right)^2}{\prod_{j=1}^{i-1} m_{k-j}} \\
&\quad - \sum_{i=2}^{p} \frac{\left(F^{-i+1-m_{k-i}}x_{k-1}\right)^2}{\prod_{j=1}^{i-1} m_{k-j}} - \sum_{i=2}^{p} \frac{\left(F^{-i+2-m_{k-i}}x_{k-1}\right)^2}{\prod_{j=1}^{i-1} m_{k-j}} \\
&\quad + \sum_{i=2}^{p} \left[\frac{\sum_{l=-i+2-m_{k-i+1}}^{-i+2-m_{k-i}} F^l x_{k-1}}{\prod_{j=1}^{i-1} m_{k-j}} \right] \\
&\quad + \sum_{i=2}^{p} \left[\frac{\sum_{\substack{l,s=-i+2-m_{k-i+1} \\ l\neq s}}^{-i+1} F^l x_{k-1} F^s x_{k-1}}{\prod_{j=1}^{i-1} m_{k-j}} \right]
\end{aligned}
\tag{C.8}
$$

Finally, the result follows by substituting $(C.8)$ into $(C.7)$. □

C.4 Algorithm and Flowchart For Simulation

The simulated estimate $y_{m_k,k}^s$ for the energy commodity model follows the Euler scheme

$$
y_{m_k,k}^s = y_{m_{k-1},k-1}^s + \hat{a}_{m_{k-1},k-1}(\hat{\mu}_{m_{k-1},k-1} - y_{m_{k-1},k-1}^s) y_{m_{k-1},k-1}^s \Delta t + \hat{\sigma}_{m_{k-1},k-1} y_{m_{k-1},k-1}^s \Delta W_{m_k,k}. \tag{C.9}
$$

Algorithm 2 Simulation scheme

Given initials r, ϵ, $\{\hat{s}^2_{m_0,0}\}_{m_0\in OS_0}$, $\{\hat{s}^2_{m_{-1},-1}\}_{m_{-1}\in OS_{-1}}$, $\{\bar{S}^2_{m_{-1},-1}\}_{m_{-1}\in OS_{-1}}$, $\{y^s_{m_0,0}\}_{m_0\in OS_0}$,

for $k = 1$ to N **do**,

  for $m_{k-1} = 2$ to $r + k - 2$ **do**,

    Compute $\hat{a}_{m_{k-1},k-1}$, $\hat{\mu}_{m_{k-1},k-1}$,

    for $m_{k-2} = 2$ to $r + k - 3$ **do**,

      Compute $\bar{S}^2_{m_{k-1},k-1}$, $\hat{s}^2_{m_k,k}$,$y^s_{m_k,k}$, Ξ_{m_k,k,y_k}

    end for

  end for

end for

if $\Xi_{m_k,k,y_k} < \epsilon$ **then**,

Save $\hat{m}_k$, $\hat{m}_{k-1}$, $\hat{m}_{k-2}$

else

  Find $\hat{m}_k$ that minimizes Ξ_{m_k,k,y_k}.

end if

Compute $a_{\hat{m}_k,k}$, $\mu_{\hat{m}_k,k}$,$s^2_{\hat{m}_k,k}$,$y^s_{\hat{m}_k,k}$.

Similar algorithm can be generated for the interest rate model.

REMARK 32 We give the first iterate for the energy commodity model.

Given initials r, ϵ, $\{s^2_{m_0,0}\}_{m_0\in OS_0}$, $\{s^2_{m_{-1},-1}\}_{m_{-1}\in OS_{-1}}$, $\{\bar{S}^2_{m_{-1},-1}\}_{m_{-1}\in OS_{-1}}$, $\{y^s_{m_0,0}\}_{m_0\in OS_0}$,

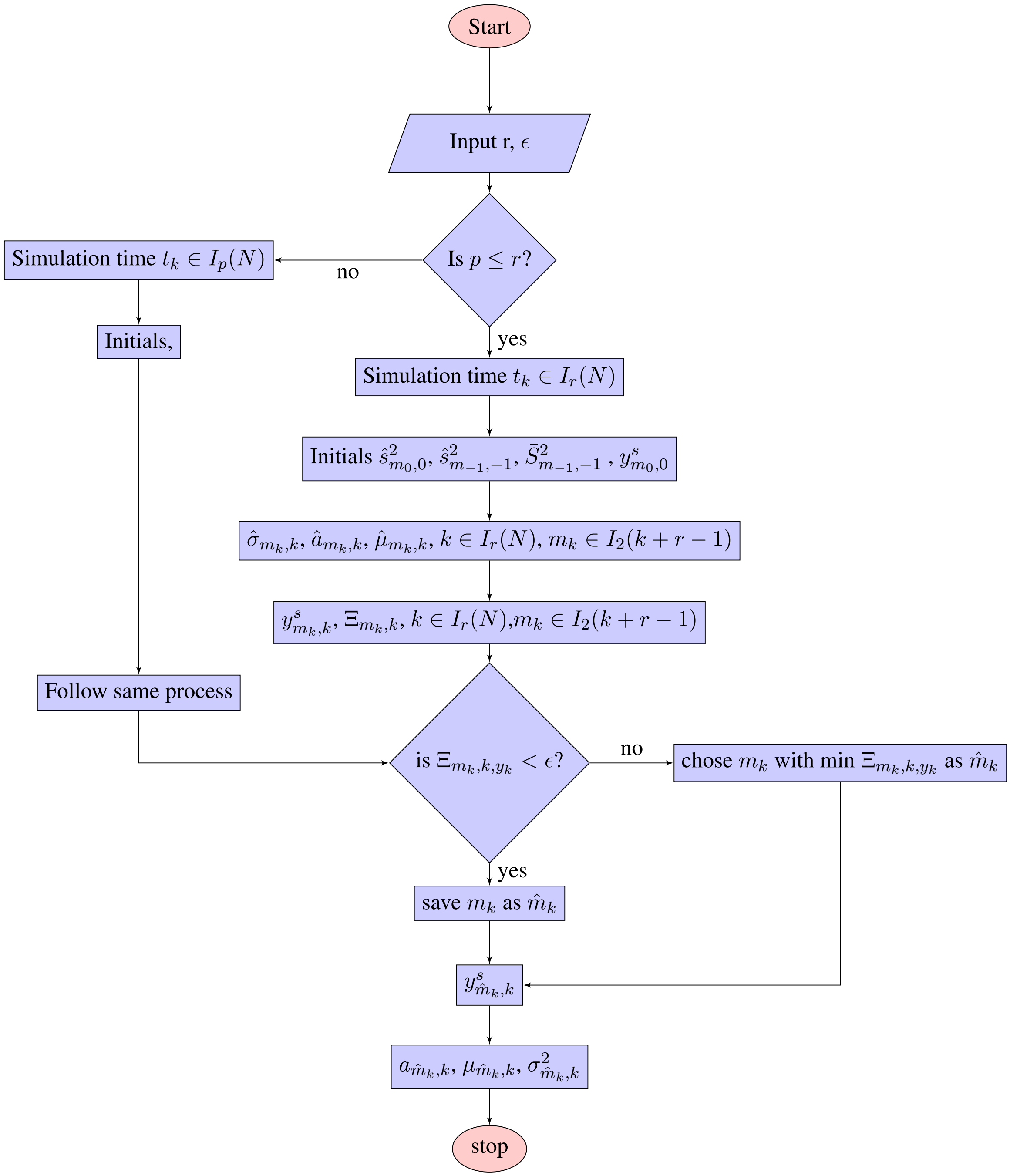
Compute $a_{m_0,0}$, $\mu_{m_0,0}$.

For k=1:

Compute $y^s_{m_1,1}$ using $(C.9)$. If $\Xi_{m_1,1,y_1} < \epsilon$, save $\hat{m}_1, \hat{m}_0, \hat{m}_{-1}$, else, find values of m_1 that minimizes $\Xi_{m_1,1,y_1}$.

Compute $a_{\hat{m}_0,0}$, $\mu_{\hat{m}_0,0}$,$s^2_{\hat{m}_1,1}$,$y^s_{\hat{m}_1,1}$.

Next, we give a flowchart similar to the algorithm above.



Flowchart 2: LLGMM Simulation Algorithm.

Appendix D

D.1 Proof of Lemma 9.3

Proof of Lemma 9.3 for small m_k, $m_{k-1} \leq m_k$, *Proof.*

$$\begin{aligned}
s^{i,j}_{m_k,k} &= \frac{1}{m_k}\left[\sum_{\iota=-m_k+1}^{0}\left(F^{\iota}x_i(k-1)\right)\left(F^{\iota}x_j(k-1)\right)\right. \\
&\quad \left. -\frac{1}{m_k}\left(\sum_{a=-m_k+1}^{0}F^{a}x_i(k-1)\right)\left(\sum_{a=-m_k+1}^{0}F^{a}x_j(k-1)\right)\right] \\
&= \frac{1}{m_k}\left[\sum_{\iota=-m_k+1}^{-m_{k-1}-1}\left(F^{\iota}x_i(k-1)\right)\left(F^{\iota}x_j(k-1)\right)+\sum_{\iota=-m_{k-1}}^{-1}\left(F^{\iota}x_i(k-1)\right)\left(F^{\iota}x_j(k-1)\right)\right. \\
&\quad \left. +\left(F^{0}x_i(k-1)\right)\left(F^{0}x_j(k-1)\right)\right]-\frac{1}{m_k^2}\sum_{a=-m_k+1}^{0}F^{a}x_i(k-1)\sum_{a=-m_k+1}^{0}F^{a}x_j(k-1) \\
&= \frac{m_{k-1}}{m_k}s^{i,j}_{m_{k-1},k-1}+\frac{m_{k-1}}{m_k}\bar{S}^{i}_{m_{k-1},k-1}\bar{S}^{j}_{m_{k-1},k-1}-\bar{S}^{i}_{m_k,k}\bar{S}^{j}_{m_k,k} \\
&\quad +\frac{\sum_{\iota=-m_k+1}^{-m_{k-1}+1}F^{\iota}x_i(k-1)F^{\iota}x_j(k-1)}{m_k} \\
&\quad +\frac{F^{0}x_i(k-1)F^{0}x_j(k-1)-F^{-m_{k-1}}x_i(k-1)F^{-m_{k-1}}x_j(k-1)}{m_k} \\
&\quad -\frac{F^{-m_{k-1}+1}x_i(k-1)F^{-m_{k-1}+1}x_j(k-1)}{m_k}.
\end{aligned}$$

Hence,

$$\begin{aligned}
s^{i,j}_{m,k} &= \tfrac{m_{k-1}}{m_k}s^{i,j}_{m_{k-1},k-1}+\tfrac{m_{k-1}}{m_k}\bar{S}^{i}_{m_{k-1},k-1}\bar{S}^{j}_{m_{k-1},k-1}-\bar{S}^{i}_{m_k,k}\bar{S}^{j}_{m_k,k} \\
&\quad +\tfrac{\sum_{\iota=-m_k+1}^{-m_{k-1}+1}F^{\iota}x_i(k-1)F^{\iota}x_j(k-1)}{m_k}+\tfrac{F^{0}x_i(k-1)F^{0}x_j(k-1)-F^{-m_{k-1}}x_i(k-1)F^{-m_{k-1}}x_j(k-1)}{m_k} \\
&\quad -\tfrac{F^{-m_{k-1}+1}x_i(k-1)F^{-m_{k-1}+1}x_j(k-1)}{m_k}.
\end{aligned} \tag{D.1}$$

Next, we find an expression connecting $s^{i,j}_{m_{k-1},k-1}$, $\bar{S}^i_{m_{k-1},k-1}\bar{S}^j_{m_{k-1},k-1}$ and $\bar{S}^i_{m_k,k}\bar{S}^j_{m_k,k}$. By definition and simplification,

$$
\begin{aligned}
m_k^2\bar{S}^i_{m_k,k}\bar{S}^j_{m_k,k} &= \sum_{\iota=-m_k+1}^{0} F^\iota x_i(k-1) \sum_{\iota=-m_k+1}^{0} F^\iota x_j(k-1) \\
&= \sum_{\iota=-m_k+1}^{0} F^\iota x_i(k-1)F^\iota x_j(k-1) + \sum_{\substack{l,s=-m_k+1 \\ l\neq s}}^{0} F^l x_i(k-1)F^s x_j(k-1) \\
&= (m_{k-1})s^{i,j}_{m_{k-1},k-1} + m_{k-1}\bar{S}^i_{m_{k-1},k-1}\bar{S}^j_{m_{k-1},k-1} + F^0 x_i(k-1)F^0 x_j(k-1) \\
&\quad -F^{-m_{k-1}}x_i(k-1))F^{-m_{k-1}}x_j(k-1) \\
&\quad -F^{-m_{k-1}+1}x_i(k-1)F^{-m_{k-1}+1}x_j(k-1) \\
&\quad +\sum_{\iota=-m_k+1}^{-m_{k-1}+1} F^\iota x_i(k-1)F^\iota x_j(k-1) \\
&\quad +\sum_{\substack{l,s=-m_k+1 \\ l\neq s}}^{0} F^l x_i(k-1)F^s x_j(k-1)
\end{aligned}
\tag{D.2}
$$

Substituting $(D.2)$ into $(D.1)$, we have

$$
\begin{aligned}
s^{i,j}_{m,k} &= \tfrac{m_k-1}{m_k}\left[\tfrac{m_{k-1}}{m_k}s^{i,j}_{m_{k-1},k-1} + \tfrac{m_{k-1}}{m_k}\bar{S}^i_{m_{k-1},k-1}\bar{S}^j_{m_{k-1},k-1}\right] \\
&\quad +\tfrac{m_k-1}{m_k}\left[\tfrac{F^0 x_i(k-1)F^0 x_j(k-1) - F^{-m_{k-1}}x_i(k-1)F^{-m_{k-1}}x_j(k-1)}{m_k}\right. \\
&\quad \left. -\tfrac{F^{-m_{k-1}+1}x_i(k-1)F^{-m_{k-1}+1}x_j(k-1)}{m_k} + \tfrac{\sum_{\iota=-m_k+1}^{-m_{k-1}+1} F^\iota x_i(k-1)F^\iota x_j(k-1)}{m_k}\right] \\
&\quad -\frac{\sum_{\substack{l,s=-m_k+1 \\ l\neq s}}^{0} F^l x_i(k-1)F^s x_j(k-1)}{m_k(m_k-1)}.
\end{aligned}
\tag{D.3}
$$

Likewise, using $(D.2)$,

$$
\begin{aligned}
m_{k-1}^2\bar{S}^i_{m_{k-1},k-1}\bar{S}^j_{m_{k-1},k-1} &= (m_{k-2})s^{i,j}_{m_{k-2},k-2} + m_{k-2}\bar{S}^i_{m_{k-2},k-2}\bar{S}^j_{m_{k-2},k-2} \\
&\quad +F^{-1}x_i(k-1)F^{-1}x_j(k-1) \\
&\quad -F^{-m_{k-2}-1}x_i(k-1)F^{-m_{k-2}-1}x_j(k-1) \\
&\quad -F^{-m_{k-2}}x_i(k-1)F^{-m_{k-2}}x_j(k-1) \\
&\quad +\sum_{\iota=-m_{k-1}}^{-m_{k-2}} F^\iota x_i(k-1)F^\iota x_j(k-1) \\
&\quad +\sum_{\substack{l,s=-m_{k-1} \\ l\neq s}}^{-1} F^l x_i(k-1)F^s x_j(k-1).
\end{aligned}
$$

Also,

$$\begin{aligned}
m_{k-2}^2\bar{S}^i_{m_{k-2},k-2}\bar{S}^j_{m_{k-2},k-2} &= (m_{k-3})s^{i,j}_{m_{k-3},k-3} + m_{k-3}\bar{S}^i_{m_{k-3},k-3}\bar{S}^j_{m_{k-3},k-3} \\
&+F^{-2}x_i(k-1)F^{-2}x_j(k-1) \\
&-F^{-m_{k-3}-2}x_i(k-1)F^{-m_{k-3}-2}x_j(k-1) \\
&-F^{-m_{k-3}-1}x_i(k-1)F^{-m_{k-3}-1}x_j(k-1) \\
&+\sum_{\iota=-m_{k-2}-1}^{-m_{k-3}-1} F^{\iota}x_i(k-1)F^{\iota}x_j(k-1) \\
&+\sum_{\substack{l,s=-m_{k-2}-1\\ l\neq s}}^{-2} F^{l}x_i(k-1)F^{s}x_j(k-1).
\end{aligned}$$

Continuing in this sense and substituting $\bar{S}^i_{m_{k-i},k-i}\bar{S}^j_{m_{k-i},k-i}$, $i = 2, ..., d-1$ into $\bar{S}^i_{m_{k-1},k-1}\bar{S}^j_{m_{k-1},k-1}$, we have

$$\begin{aligned}
(m_{k-1})\bar{S}^i_{m_{k-1},k-1}\bar{S}^j_{m_{k-1},k-1} &= \sum_{\iota=2}^{d}\left[\frac{m_{k-i}}{\prod_{a=1}^{\iota-1} m_{k-j}}\right] s^{i,j}_{m_{k-i},k-i} + \frac{m_{k-d}}{\prod_{a=1}^{d-1} m_{k-j}}\bar{S}^i_{m_{k-d},k-d}\bar{S}^j_{m_{k-d},k-d} \\
&+\sum_{\iota=2}^{d}\frac{F^{-\iota+1}x_i(k-1)F^{-\iota+1}x_j(k-1)}{\prod_{a=1}^{\iota-1} m_{k-j}} \\
&-\sum_{\iota=2}^{d}\frac{F^{-\iota+1-m_{k-i}}x_i(k-1)F^{-\iota+1-m_{k-i}}x_j(k-1)}{\prod_{a=1}^{\iota-1} m_{k-j}} \\
&-\sum_{\iota=2}^{d}\frac{F^{-i+2-m_{k-i}}x_i(k-1)F^{-i+2-m_{k-i}}x_j(k-1)}{\prod_{a=1}^{\iota-1} m_{k-j}} \\
&+\sum_{\iota=2}^{d}\left[\frac{\sum_{l=-i+2-m_{k-i+1}}^{-i+2-m_{k-i}} F^{l}x_i(k-1)F^{l}x_j(k-1)}{\prod_{a=1}^{\iota-1} m_{k-j}}\right] \\
&+\sum_{\iota=2}^{d}\left[\frac{\sum_{\substack{l,s=-i+2-m_{k-i+1}\\ l\neq s}}^{-i+1} F^{l}x_i(k-1)F^{s}x_j(k-1)}{\prod_{a=1}^{\iota-1} m_{k-j}}\right]
\end{aligned} \tag{D.4}$$

Finally, the result follows by substituting $(D.4)$ into $(D.3)$.

□

D.2 Proof of Lemma 9.3 for small m_k

Proof of Lemma 9.3 for small m_k, $m_k \leq m_{k-1}$, *Proof.* Following the same steps, if $m_k \leq m_{k-1}$,

$$\begin{cases} s^{i,j}_{m_k,k} = \frac{m_k-1}{m_k}\left[\sum_{\iota=1}^{d}\left[\frac{m_{k-i}}{\prod_{a=0}^{\iota-1} m_{k-j}}\right] s^{i,j}_{m_{k-i},k-i} + \frac{m_{k-d}}{\prod_{a=0}^{d-1} m_{k-j}}\bar{S}^{i}_{m_{k-d},k-d}\bar{S}^{j}_{m_{k-d},k-d}\right] \\ \quad +\varpi^{i,j}_{m_{k-1},k-1},\; m_k \leq m_{k-1} \\ \varpi^{i,j}_{m_{k-1},k-1} = \frac{m_k-1}{m_k}\left[\sum_{\iota=1}^{d}\frac{F^{-i+1}x_i(k-1)F^{-i+1}x_j(k-1)}{\prod_{a=0}^{\iota-1} m_{k-j}} - \sum_{\iota=1}^{d}\left[\frac{\sum_{l=-i+1-m_{k-i}}^{-i+1-m_{k-i+1}} F^l x_i(k-1)F^l x_j(k-1)}{\prod_{a=0}^{\iota-1} m_{k-j}}\right]\right. \\ \quad \left. + \sum_{\iota=1}^{d}\left[\frac{\sum_{\substack{l,s=-i+2-m_{k-i+1} \\ l\neq s}}^{-i+1} F^l x_i(k-1)F^s x_j(k-1)}{\prod_{a=0}^{\iota-1} m_{k-j}}\right]\right] \\ \quad -\frac{1}{m_k}\sum_{\substack{l,s=-m_k+1 \\ l\neq s}}^{0} F^l x_i(k-1)F^s x_j(k-1), \end{cases}$$

□

D.3 Proof of Lemma 9.3 for large m_k

Proof of Lemma 9.3 for large m_k

Proof.

$$\begin{aligned} s^{i,j}_{m_k,k} &= \frac{1}{m_k-1}\left[\sum_{\iota=-m_k+1}^{0}\left(F^{\iota}x_i(k-1)\right)\left(F^{\iota}x_j(k-1)\right)\right. \\ &\quad \left. -\frac{1}{m_k}\left(\sum_{a=-m_k+1}^{0} F^a x_i(k-1)\right)\left(\sum_{a=-m_k+1}^{0} F^a x_j(k-1)\right)\right] \\ &= \frac{1}{m_k-1}\left[\sum_{\iota=-m_{k-1}}^{-1} F^{\iota}x_i(k-1)F^{\iota}x_j(k-1)\right. \\ &\quad -\frac{1}{m_{k-1}}\sum_{\iota=-m_{k-1}}^{-1} F^{\iota}x_i(k-1)\sum_{\iota=-m_{k-1}}^{-1} F^{\iota}x_j(k-1) \\ &\quad \left. +\frac{1}{m_{k-1}}\sum_{\iota=-m_{k-1}}^{-1} F^{\iota}x_i(k-1)\sum_{\iota=-m_{k-1}}^{-1} F^{\iota}x_j(k-1)\right] \\ &\quad +\frac{1}{m_k-1}\left[F^0x_i(k-1)F^0x_j(k-1) - F^{-m_{k-1}}x_i(k-1)F^{-m_{k-1}}x_j(k-1)\right. \\ &\quad \left. -F^{-m_{k-1}+1}x_i(k-1)F^{-m_{k-1}+1}x_j(k-1)\right] \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{m_k-1}\left[\sum_{\iota=-m_k+1}^{-m_{k-1}+1}F^\iota x_i(k-1)\sum_{\iota=-m_k+1}^{-m_{k-1}+1}F^\iota x_j(k-1)\right.\\
&\left.-\frac{1}{m_k}\sum_{\iota=-m_k+1}^{0}F^\iota x(k-1)\sum_{\iota=-m_k+1}^{0}F^\iota x(k-1)\right]\\
= \;& \frac{m_{k-1}-1}{m_k-1}s^{i,j}_{m_{k-1},k-1}+\frac{m_{k-1}}{m_k-1}\bar{S}^i_{m_{k-1},k-1}\bar{S}^j_{m_{k-1},k-1}-\frac{m_k}{m_k-1}\bar{S}^i_{m_k,k}\bar{S}^j_{m_k,k}\\
&+\frac{F^0x_i(k-1)F^0x_j(k-1)-F^{-m_{k-1}}x_i(k-1)F^{-m_{k-1}}x_j(k-1)}{m_k-1}\\
&-\frac{F^{-m_{k-1}+1}x_i(k-1)F^{-m_{k-1}+1}x_j(k-1)}{m_k-1}+\frac{\sum\limits_{\iota=-m_k+1}^{-m_{k-1}+1}F^\iota x_i(k-1)F^\iota x_j(k-1)}{m_k-1}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
s^{i,j}_{m,k} = \;& \frac{m_{k-1}-1}{m_k-1}s^{i,j}_{m_{k-1},k-1}+\frac{m_{k-1}}{m_k-1}\bar{S}^i_{m_{k-1},k-1}\bar{S}^j_{m_{k-1},k-1}-\frac{m_k}{m_k-1}\bar{S}^i_{m_k,k}\bar{S}^j_{m_k,k}\\
&+\frac{F^0x_i(k-1)F^0x_j(k-1)-F^{-m_{k-1}}x_i(k-1)F^{-m_{k-1}}x_j(k-1)-F^{-m_{k-1}+1}x_i(k-1)F^{-m_{k-1}+1}x_j(k-1)}{m_k-1}\\
&+\frac{\sum\limits_{\iota=-m_k+1}^{-m_{k-1}+1}F^\iota x_i(k-1)F^\iota x_j(k-1)}{m_k-1}.
\end{aligned}
\tag{D.5}
$$

Next, we find an expression connecting $\bar{S}^i_{m_k,k}\bar{S}^j_{m_k,k}$, $\bar{S}^i_{m_k-1,k-1}\bar{S}^j_{m_k-1,k-1}$ and $s^{i,j}_{m_k-1,k-1}$. By definition and simplification,

$$
\begin{aligned}
m_k^2\bar{S}^i_{m_k,k}\bar{S}^j_{m_k,k} = \;& \sum_{\iota=-m_k+1}^{0}F^\iota x_i(k-1)\sum_{\iota=-m_k+1}^{0}F^\iota x_j(k-1)\\
= \;& \sum_{\iota=-m_k+1}^{0}F^\iota x_i(k-1)F^\iota x_j(k-1)\\
&+\sum_{\substack{l,s=-m_k+1\\ l\neq s}}^{0}F^l x_i(k-1)F^s x_j(k-1)\\
= \;& (m_{k-1}-1)s^{i,j}_{m_{k-1},k-1}+m_{k-1}\bar{S}^i_{m_{k-1},k-1}\bar{S}^j_{m_{k-1},k-1}\\
&+F^0x_i(k-1)F^0x_j(k-1)-F^{-m_{k-1}}x_i(k-1)F^{-m_{k-1}}x_j(k-1)\\
&-F^{-m_{k-1}+1}x_i(k-1)F^{-m_{k-1}+1}x_j(k-1)\\
&+\sum_{\iota=-m_k+1}^{-m_{k-1}+1}F^\iota x_i(k-1)F^\iota x_j(k-1)\\
&+\sum_{\substack{l,s=-m_k+1\\ l\neq s}}^{0}F^l x_i(k-1)F^s x_j(k-1)
\end{aligned}
\tag{D.6}
$$

Substituting ($D.6$) into ($D.5$), we have

$$
\begin{aligned}
s^{i,j}_{m,k} \quad = \quad & \tfrac{m_{k-1}-1}{m_k} s^{i,j}_{m_{k-1},k-1} + \tfrac{m_{k-1}}{m_k}\bar{S}^i_{m_{k-1},k-1}\bar{S}^j_{m_{k-1},k-1} \\
& + \tfrac{F^0 x_i(k-1) F^0 x_j(k-1) - F^{-m_{k-1}} x_i(k-1) F^{-m_{k-1}} x_j(k-1) - F^{-m_{k-1}+1} x_i(k-1) F^{-m_{k-1}+1} x_j(k-1)}{m_k} \\
& + \tfrac{\sum\limits_{\iota=-m_k+1}^{-m_{k-1}+1} F^\iota x_i(k-1) F^\iota x_j(k-1)}{m_k} - \tfrac{\sum\limits_{\substack{l,s=-m_k+1 \\ l\neq s}}^{0} F^l x_i(k-1) F^s x_j(k-1)}{m_k(m_k-1)}.
\end{aligned}
\tag{D.7}
$$

Likewise,

$$
\begin{aligned}
m^2_{k-1}\bar{S}^i_{m_{k-1},k-1}\bar{S}^j_{m_{k-1},k-1} \quad = \quad & (m_{k-2}-1)s^{i,j}_{m_{k-2},k-2} + m_{k-2}\bar{S}^i_{m_{k-2},k-2}\bar{S}^j_{m_{k-2},k-2} \\
& + F^{-1}x_i(k-1)F^{-1}x_j(k-1) \\
& - F^{-m_{k-2}-1}x_i(k-1)F^{-m_{k-2}-1}x_j(k-1) \\
& - F^{-m_{k-2}}x_i(k-1)F^{-m_{k-2}}x_j(k-1) \\
& + \sum_{\iota=-m_{k-1}}^{-m_{k-2}} F^\iota x_i(k-1)F^\iota x_j(k-1) \\
& + \sum_{\substack{l,s=-m_{k-1} \\ l\neq s}}^{-1} F^l x_i(k-1)F^s x_j(k-1),
\end{aligned}
$$

$$
\begin{aligned}
m^2_{k-2}\bar{S}^i_{m_{k-2},k-2}\bar{S}^j_{m_{k-2},k-2} \quad = \quad & (m_{k-3}-1)s^{i,j}_{m_{k-3},k-3} + m_{k-3}\bar{S}^i_{m_{k-3},k-3}\bar{S}^j_{m_{k-3},k-3} \\
& + F^{-2}x_i(k-1)F^{-2}x_j(k-1) \\
& - F^{-m_{k-3}-2}x_i(k-1)F^{-m_{k-3}-2}x_j(k-1) \\
& - F^{-m_{k-3}-1}x_i(k-1)F^{-m_{k-3}-1}x_j(k-1) \\
& + \sum_{\iota=-m_{k-2}-1}^{-m_{k-3}-1} F^\iota x_i(k-1)F^\iota x_j(k-1) \\
& + \sum_{\substack{l,s=-m_{k-2} \\ l\neq s}}^{-2} F^l x_i(k-1)F^s x_j(k-1).
\end{aligned}
$$

Continuing in this sense and substituting $\bar{S}_{m_{k-i},k-i}$, $i = 2, ..., d-1$ into $\bar{S}_{m_{k-1},k-1}$, we have

$$
\begin{aligned}
(m_{k-1})\bar{S}^{i,j}_{m_{k-1},k-1} \quad = \quad & \sum_{\iota=2}^{d}\left[\tfrac{m_{k-i}-1}{\prod\limits_{a=1}^{\iota-1} m_{k-j}}\right] s^{i,j}_{m_{k-i},k-i} + \tfrac{m_{k-d}}{\prod\limits_{a=1}^{d-1} m_{k-j}}\bar{S}^i_{m_{k-d},k-d}\bar{S}^j_{m_{k-d},k-d} \\
& + \sum_{\iota=2}^{d} \tfrac{F^{-i+1}x_i(k-1)F^{-i+1}x_j(k-1)}{\prod\limits_{a=1}^{\iota-1} m_{k-j}} \\
& - \sum_{\iota=2}^{d} \tfrac{F^{-i+1-m_{k-i}}x_i(k-1)F^{-i+1-m_{k-i}}x_j(k-1)}{\prod\limits_{a=1}^{\iota-1} m_{k-j}}
\end{aligned}
\tag{D.8}
$$

$$
- \sum_{\iota=2}^{d} \frac{F^{-i+2-m_{k-i}} x_i(k-1) F^{-i+2-m_{k-i}} x_j(k-1)}{\prod_{a=1}^{\iota-1} m_{k-j}} + \sum_{\iota=2}^{d} \left[\frac{\sum_{l=-i+2-m_{k-i+1}}^{-i+2-m_{k-i}} F^l x_i(k-1) F^l x_j(k-1)}{\prod_{a=1}^{\iota-1} m_{k-j}} \right]
$$

$$
+ \sum_{\iota=2}^{d} \left[\frac{\sum_{\substack{l,s=-i+2-m_{k-i+1} \\ l \neq s}}^{-i+1} F^l x_i(k-1) F^s x_j(k-1)}{\prod_{a=1}^{\iota-1} m_{k-j}} \right] \tag{D.9}
$$

Finally, the result follows by substituting $(D.8)$ into $(D.7)$. $\square$

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